MA 3257 LECTURE 1

Fundamentals of Numerical Methods

Q: What are Numerical Methods?
A: They are **algorithms** to **approximate** the solution to problems in **continuous** mathematics.

(1) Algorithms
An algorithm is a **finite** sequence of steps or a formal procedure that a computer program implements.
  (a) Computer programs are always finite.
  (b) Approximate
  Most of the time, the exact solution to a problem is unavailable.
  (c) Continuous
  There is more to the definition of continuity. Here, it means that the solutions to the mathematical problems are **real numbers**, not integers. Real numbers are in some sense "infinite", as opposed to discrete/finite like the algorithms.

Example. Algebraic equations:

(1) $x^2 - 1 = 0$ has exact solutions $x = \pm 1$.
(2) $x^2 + 2x - 7 = 0$ has exact solutions via the quadratic formula

$$x = \frac{-2 \pm \sqrt{4 - 4\,(1)\,(-7)}}{2} = 1 \pm 2\sqrt{2},$$

but $\sqrt{2}$ is a real number, not an integer. An **approximation** is needed.
(3) $f(x) = x^5 + 2x - 7 = 0$ has no easy formula for the roots. We need **rooting-finding** algorithms to find all points $x \in \mathbb{R}$ such that $f(x) = 0$.

Example. Integrals. Find the area under f for $0 \leq x \leq 1$.

(1) Let $f(x) = e^x$.

$$\int_0^1 e^x dx = e - 1.$$

But, e is an irrational number, which requires an **approximation**.
  (a)

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

  (b)

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

(2) Let $f(x) = e^{-x^2}$. Then, $\int_0^1 e^{-x^2} dx$ is impossible since e^{-x^2} does not have an antiderivative expressed by an elementary function. Thus, we approximate it by a numerical integration scheme, such as left point rule, right point, or a little better, trapezoidal (among many numerical integration methods).

Example. Ordinary Differential Equations (ODEs).

(1) Linear ODE
Suppose we have an initial value problem, $y(0) = y_0$, and $\frac{dy}{dx} = ky$. An exact solution is

$$y(x) = y_0 e^{ky}.$$

However, the computer will still suffer from the approximation of e.

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(2) Nonlinear ODE
$$\frac{dy}{dx} = \frac{\sin y}{y} + y^2 x = f(x, y(x)), \quad y(x_0) = y_0.$$
which has no closed form solution. We need a numerical method to find $y(x)$.
$$y(x) = y_0 + \int_{x_0}^{x} f(x, y(x))\, dx$$
which asks for a good numerical integration method.

Example. Numerical Differentiation – Approximation of derivatives.
$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$
where the limit is taken by shrinking the distance from an arbitrary point z to the target point x. This "shrinking" can be done when $|z - x|$ is small enough, but not zero.

Example. Interpolation of Data: given some experimental data, can we cook up a function that pass through all of them? Then, we will use this function to make estimates where data is unavailable.

Standards of Algorithms

(1) Accuracy: for a given amount of time, how big is the error?
(2) Speed: for a given accuracy level, which algorithm is faster?
(3) Stability: Can the method fail to yield the correct answer?

To understand the accuracy, speed and stability of an algorithm (and then possibly improve them), we study them **mathematically**. This practice is commonly known as **numerical analysis**. Coding up the algorithm and having a computer running it is the **implementation** of the algorithm.

Calculus Review

(1) A function $f(x)$ is continuous at some point $x = a$ if
$$\lim_{x \to a} f(x) = f(a).$$
This is a deeper statement than it appears. The limit itself includes both sides of the limit. The LHS is a limiting value, while the RHS is the function value – these two values are NOT necessarily the same. They are the same when the function is continuous.
(2) Intermediate value theorem. If f is continuous on $[a, b]$ and $f(a) \leq k \leq f(b)$, then there exists a point $c \in [a, b]$ such that $f(c) = k$.

Notations

Symbol	Meaning
$\mathbb{R}$	real numbers
$\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$	n-tuple of real numbers, $(a_1, a_2, \ldots, a_n)$.
$\mathbb{Z}$	integers
$\mathbb{Z}^+, \mathbb{N}$	positive integers (natural numbers)
$f : X \to Y$	A function that maps from a set X to another set Y
$C^0([a, b])$	Continuous functions on $[a, b]$
$C^1([a, b])$	Functions with continuous 1st derivative on $[a, b]$
$C^k([a, b])$	Functions with continuous derivatives up to order k on $[a, b]$
$C^\infty([a, b])$	Functions with continuous derivatives up to arbitrary order on $[a, b]$