MA 3257 – SPRING 2023 C-TERM
HOMEWORK VI (DUE FEB 28TH, 2023)

Problem 1. In lecture, we've learned how to estimate λ_1, the dominant eigenvalue of a matrix A, and its corresponding eigenvector $\boldsymbol{v}_1$, via the Power Method.

Let $\left(\mu^{(m)}, \boldsymbol{v}^{(m)}\right)$ be the approximation of $(\lambda_1, \boldsymbol{v}_1)$ at the m^{th} step of the algorithm. For the following two matrices,

(1) (5 points) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, with initial guess $\boldsymbol{v}^{(0)} = (1, -1, 2)^{\mathrm{T}}$. $\lambda_1 = 4$ with $\boldsymbol{v}_1 = (1, 1, 1)^{\mathrm{T}}$. $\lambda_2 = 1$.

(2) (5 points) $A = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$, with initial guess $\boldsymbol{v}^{(0)} = (1, -2, 0, 3)^{\mathrm{T}}$. $\lambda_1 \approx 5.2361$ with $\boldsymbol{v}_1 = (1, 0.6180, 0.1180, 0.5000)^{\mathrm{T}}$. $\lambda_2 = 3.6180$.

Do the following:

- (Hand-written, 2 points) Compute the first three iterations obtained by the Power method (you are welcome to follow the Illustration in Section 9.3 before Algorithm 9.1).
- (Coding, 3 points) Confirm numerically, using the provided code power_method.m, that

$$\left|\mu^{(k)} - \lambda_1\right| \leq C \left|\frac{\lambda_2}{\lambda_1}\right|^k,$$

$$\left\|\boldsymbol{v}^{(k)} - \boldsymbol{v}_1\right\|_\infty \leq D \left|\frac{\lambda_2}{\lambda_1}\right|^k,$$

  by producing first
  - A table

k	$\mu^{(k)}$	$\boldsymbol{v}^{(k)}$	$\left\|\mu^{(k)} - \lambda_1\right\|$	$\left\|\boldsymbol{v}^{(k)} - \boldsymbol{v}_1\right\|_\infty$
0	N/A	$\boldsymbol{v}^{(0)}$	N/A	

  - Semilogy plots of

$$\left\|\boldsymbol{v}^{(k)} - \boldsymbol{v}_1\right\|_\infty \text{ v.s. } k$$

    and

$$\left|\mu^{(k)} - \lambda_1\right| \text{ v.s. } k$$

    Doing semilogy plots extracts the power from an exponential relationship, such as the one from above. For more information on this type of plots, feel free to read its Wikipedia page.

This practice is what consistutes the beginning of a convergence study. It certainly requires some guidance from pen-and-paper work, such as the proof of knowing that the rate of convergence involves $\left|\frac{\lambda_2}{\lambda_1}\right|$, among many other relevant facts about your problem.

Problem 2. (5 points) Use the Bisection method to find p_3 for $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$. (Bonus 3 points if you write a program to print more iterates and show convergence).

Problem 3. (Fixed-point Iteration: 10 points)

- (4 points) Use algebraic manipulation to show that each of the following functions has a fixed point at p precisely when $f(p) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$.
  (1) $g_1(x) = \left(3 + x - 2x^2\right)^{1/4}$;
  (2) $g_2(x) = \left(\frac{x+3-x^4}{2}\right)^{1/2}$;
  (3) $g_3(x) = \left(\frac{x+3}{x^2+2}\right)^{1/2}$;
  (4) $g_4(x) = \frac{3x^4+2x^2+3}{4x^3+4x-1}$.
- (4 points) Perform four iterations, if possible, on each of the functions $g_i(x)$, $i = 1, 2, 3, 4$. Let $p_0 = 1$ and $p_{n+1} = g(p_n)$ for $n = 0, 1, 2, 3$.
  - (2 points) Which function do you think gives the best approximation to the solution?
- (Extra Credit 5 points) Numerically justify your answer for the last question (which function is the best), by plotting in a similar fashion as in Problem 1 (convergence study).