MA 3257 – SPRING 2023 C-TERM
HOMEWORK V (DUE FEB 20TH, 2023)

Problem 1. (5 points) Consider

$$A = \begin{bmatrix} 3.9 & 1.6 \\ 7.8 & 3.3 \end{bmatrix}.$$

- (2 points) Compute its condition number relative to $\|\cdot\|_\infty$, i.e. $\kappa(A) = \|A\|_\infty \left\|A^{-1}\right\|_\infty$. This involves knowing how to invert a matrix. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$, we have

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- (1 points) Interpret the magnitude of the condition number by considering the linear dependence of the columns of the matrix.
- (2 point) For $\boldsymbol{b} = (1, 2)^{\mathrm{T}}$, provide an approximate solution $\tilde{\boldsymbol{x}}$ to $A\boldsymbol{x} = \boldsymbol{b}$ using naive Gaussian elimination with 3-digit rounding, and compute the residual vector $\boldsymbol{r} = \boldsymbol{b} - A\tilde{\boldsymbol{x}}$. Does $\kappa(A)$ satisfy the inequality (Theorem 7.27 in textbook)

$$\frac{\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|_\infty}{\|\boldsymbol{x}\|_\infty} \le \kappa(A) \frac{\|\boldsymbol{r}\|_\infty}{\|\boldsymbol{b}\|_\infty}?$$

Problem 2. (10 points) Implement the Jacobi's iterative method per Algorithm 7.1 in the textbook. Please follow the **component formulation of the method**,

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j \ne i} a_{ij} x_j^{(k)} \right), \quad i = 1, 2, \ldots, n.$$

Test your results on the system

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 7, \\ x_1 + x_2 + x_3 &= 2, \\ 2x_1 + 2x_2 + x_3 &= 5, \end{aligned}$$

with an initial guess of $\boldsymbol{x}^{(0)} = (0, 0, 0)^{\mathrm{T}}$. The system has a true solution $\boldsymbol{x} = (1, 2, -1)^{\mathrm{T}}$. Use the following checkpoints.

(1) For each iteration, save the approximation $\boldsymbol{x}^{(k)}$.
(2) Compute the residual vector $\boldsymbol{r}^{(k)} = \boldsymbol{b} - A\boldsymbol{x}^{(k)}$.
(3) Compute $\left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}\right\|_\infty$ and $\left\|\boldsymbol{r}^{(k)}\right\|_\infty$.
(4) Form the following table:

k	$\boldsymbol{x}^{(k)}$	$\left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}\right\|_\infty$	$\frac{\left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}\right\|_\infty}{\left\|\boldsymbol{x}^{(k)}\right\|_\infty}$	$\boldsymbol{r}^{(k)}$	$\left\|\boldsymbol{r}^{(k)}\right\|_\infty$
0	$(0,0,0)$	2	N/A	$(7,2,5)$	7
1	.	.	$\frac{\left\|\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}\right\|_\infty}{\left\|\boldsymbol{x}^{(1)}\right\|_\infty}$	.	.
2	.	.	.	.	.
3	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.

(5) Among column 3, 4 and 6, we have three measures of convergence, namely,

$$\left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}\right\|_\infty, \quad \frac{\left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}\right\|_\infty}{\left\|\boldsymbol{x}^{(k)}\right\|_\infty}, \quad \left\|\boldsymbol{r}^{(k)}\right\|_\infty.$$

Fix a tolerance level of $\epsilon = 10^{-5}$. Find out at which iteration k such that the convergence is achieved under each measurement. Discuss.

Problem 3. (10 points) Implement the Gauss-Seidel Iterative Method (Algorithm 7.2) by modifying the Jacobi iterative scheme from Problem 2. Please follow the **component formulation of the method**,

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right], \quad i = 1, 2, \dots n.$$

Produce a table of inputs and outputs like Step 4. Discuss.

(Extra credit, 5 points) How many iterations did Gauss-Seidel take to converge? Why did it take so long/short?