MA 3257 – SPRING 2023 C-TERM
HOMEWORK II (DUE JAN 27TH, 2023)

This assignment contains some verification of results derived in lecture and a few MATLAB coding exercises to get you up to speed for later sections.

Problem 1. (Stable Function Evaluation: (1+2+2) 5 points) In class, we have shown that evaluating the function $f(x) = \sqrt{x+1} - 1$ is a well-conditioned problem (on paper). However, direct evaluation using a machine via the three steps (addition, square root, subtraction) shows that it is unstable (as shown in class).

As a remedy,

(1) Perform the conjugation technique to f and write down the sequence of operations;
(2) Investigate the condition number of each operation;
(3) Show that this revised evaluation algorithm is stable (meaning that **all** steps are well-conditioned).

Problem 2. (Absolute condition number: (2+3) 5 points) In Lecture 5 and 6, we saw the definition of condition number κ as

$$\mathcal{E}_{\text{output}} \approx \kappa \mathcal{E}_{\text{input}}$$

where $\mathcal{E}$ is taken to be the relative error. In this case, we call κ_{rel} **the relative condition number**.

Now, consider $\mathcal{E}$ under the setting of the **absolute error**, i.e., given a perturbation $\delta \in \mathbb{R}$ to some input x, the absolute error is simply the perturbation size $|\delta|$ (coming from $|x - (x+\delta)|$, as opposed to relative error, $\left|\frac{x-(x+\delta)}{x}\right| = \left|\frac{\delta}{x}\right|$). Denote κ_{abs} as the **absolute condition number**.

(1) Derive an approximate formula for κ_{abs} for the problem of function evaluation $f(x)$ at $x = x_0$ where f is continuously differentiable.
(2) Consider the function $f(x) = x^p$ where $p \geq 0$. Find κ_{abs} and κ_{rel} for evaluating f at an arbitrary point $x = x_0$. Provide, for each error setting, some conditions on p and x_0 such that the evaluation problem of f is well-conditioned (say, $\kappa \leq 1$).

Problem 3. (5 points) Find the rate of convergence of the following sequences as $n \to \infty$:

(1) (2 points) $\lim_{n\to\infty} [\ln(n+1) - \ln(n)] = 0$.
(2) (3 points) Consider $\alpha_n = \frac{2n^2+4n}{n^2+2n+1}$ for $n = 1, 2, \ldots$.
Show that $\alpha_n = \alpha + O\left(\frac{1}{n^2}\right)$ where $\alpha = \lim_{n\to\infty} \alpha_n$. This proves that α_n converges to α quadratically/at 2nd order.
(3) (Bonus, 3 points, 1 each)
  (a) $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$.
  (b) $\lim_{n\to\infty} \sin\left(\frac{1}{n^2}\right) = 0$.
  (c) $\lim_{n\to\infty} \left(\sin\left(\frac{1}{n}\right)\right)^2 = 0$.

Problem 4. (Coding exercise: Matrix-vector multiplication (5 points))

Write a MATLAB function that achieves the following:

Input: A, an $m \times n$ matrix; b, an $k \times 1$ column vector.

Output: If A and b are dimensionally compatible for matrix-vector multiplication, we have $y = Ab$, an $m \times 1$ column vector that satisfies

$$y_i = \sum_{j=1}^{n} A_{ij} b_j, \quad i = 1, \ldots, m.$$

You must consider any possible user inputs, and print out error messages if the dimension compatibility between the matrix and the vector is violated. You may use the matlab built-in function `size()` that returns a vector that yields the number of rows and columns of the input.

Test your program using

```
A=rand(m,n);
b=rand(k,1);
y=matrix_vector_m(A,b);
y_matlab=A*b;%MATLAB␣built-in
err=norm(y-y_matlab,2);
```

where m, n and k are completely arbitrary choices – that is, if the dimensions are compatible, then this program carries through; if not, then an error message appears. **Print out test results that showcase the robustness of your program**.

Remark. The measure of error used here is called l^2-error, that is, square of sum of square differences between the entries of the two vectors. Given $v \in \mathbb{R}^n$, its l^2-norm is given by

$$\|v\|_2 = \sqrt{\sum_{i=1}^{n} v_i^2}.$$

Problem 5. (Gaussian elimination with back substitution: 5 points) Use two-digit rounding and Gaussian elimination with back substitution to solve the following system of equations

$$x_1 + 2x_2 + x_3 = 5,$$
$$\frac{3}{17}x_1 + \frac{2}{17}x_2 + \frac{4}{17}x_3 = 1,$$
$$4x_1 + 4x_2 + 3x_3 = 26,$$

and compare to the true solution $x_1 = 9, x_2 = -1, x_3 = -2$. (You may use the l^2-error from the last problem, in an absolute or relative sense, your choise).