MATH 3257 C01
Spring 2023 C Term
Exam 2
03/03/23
Time Limit: 50 Minutes

Name (Print): SOLUTION KEY

This exam contains 7 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books and notes. You may use a **scientific calculator** on this exam. You are allowed **one page (double-sided)** of notes.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- Manage the empty space wisely.
- Do NOT write in the table on the right.

Problem	Points	Score
1	20	
2	30	
3	30	
4	20	
Total:	100	

1. (20 points) True or False. Each problem has two components: the answer and the reason/correction, each worth 2 points. If true, state a concise reason or quote a theorem/homework. If false, give a counterexample or correct the statement by pointing out the logical fallacies (via some known results/theorems).

   (a) The Gauss-Seidel method is guaranteed to fail (to converge) when used to solve a system $Ax = b$ if A is not diagonally dominant.

   Solution: False. Gauss-Seidel may still converge for matrices that are not diagonally dominant. Being diagonally dominant is only a sufficient but not necessary condition for GS to converge.

   (b) A function $g(x)$ has a fixed point on the bounded interval $[a, b]$ if the function value satisfies $a \leq g(x) \leq b$ for every x that lies in $[a, b]$.

   Solution: Technically, false, because we need to g to be continuous also.

   Credits are also given to those who answered True by quoting the Existence Theorem. This is NOT by definition of a fixed point.

   (c) The Method of Successive Over-Relaxation (SOR) is an independent method, completely different from Gauss-Seidel and Jacobi iteration.

   Solution: False. SOR is a continuation of the GS method by adding a multiplier that controls update size.

   (d) The Power Method for estimating the dominant eigenvalue of a matrix converges very fast if the largest and the second largest eigenvalue are very close in magnitude.

   Solution: False. The rate of convergence depends on $\left|\frac{\lambda_2}{\lambda_1}\right|$, which means the further apart λ_2 and λ_1 are, the faster the convergence.

   (e) Applying the Secant Method to solve the root-finding problem $f(x) = 0$ needs the derivative of f.

   Solution: False. The very creation of the Secand Method is to **avoid** derivative information. It replaces the derivative with the slope of the secant line.

2. (30 points) Consider the root-finding problem

$$f(x) = x^3 - 2x + 1 = 0.$$

(a) (10 points) Use the Intermediate Value Theorem to provide a reasonable interval where a root is guaranteed to exist.

Solution: First note that $f(x)$ is a continuous function. We also find that

$$f(0) = 1 > 0, \quad f(-2) = -3 < 0.$$

By Intermediate Value Theorem, there exists a point $c \in [-2, 0]$ such that $f(c) = 0$.

Note that the answer is not unique. You may find other intervals that also contain a root.

(b) (5 points) (Bisection) Use the Bisection method on this interval, and produce 3 iterates, b_1, b_2, b_3. Evaluate $f(b_i)$ for $i = 1, 2, 3$. Is the value getting closer to zero?

Solution: Note that the answers can be different from what you have, depending on the interval you found. The idea is the same.

Using $[-2, 0]$ as our search space, we note the midpoint is $b_1 = -1$ and hence evaluate f at this point and find,

$$f(-1) = 2 > 0.$$

This tells us that the root is in $[-2, -1]$ because $f(-1)$ has opposite sign as $f(-2)$. The midpoint of $[-2, -1]$ is $b_2 = -3/2$ that evaluates to

$$f(-3/2) = 5/8 > 0.$$

This has opposite sign as $f(-2)$, which prompts us to search $[-2, -3/2]$, at the middle point $b_3 = -7/4$. Indeed,

$$f(-7/4) = -55/64.$$

We observe that even though we are narrowing down where the root is, the values $f(b_i)$ aren't getting closer to zero (as quickly as we think). But that's ok. This at least tells us that the root is in $\left[-\frac{7}{4}, -\frac{3}{2}\right]$.

(c) (5 points) (Fixed-point Iteration) Propose a function $g(x)$ such that its fixed-point satisfies $f(x) = 0$.

Solution: Again, the answers here may not be unique. You can propose infinitely many such $g(x)$'s. Some candidates would be

$$g(x) = x \pm f(x), (2x-1)^{\frac{1}{3}}, \frac{x^3+1}{2}, etc.$$

The main idea is to isolate x from $f(x) = 0$ by whichever way you feel.

However, choosing g such that a fixed point exists takes experience because you want g to guarantee a unique fixed point in your search interval.

(d) (5 points) (Fixed-point Iteration) Pick any initial guess p_0 that lies in the interval you found from part (a). Iterate $p_n = g(p_{n-1})$ 3 times, i.e., obtain p_1, p_2, p_3.

Solution: You pick whatever you want from the interval you used in part (a). In fact, you are free to pick any point from the interval that you have narrowed down to using the Bisection method.

For practical uses, you may choose $p_0 = -1.6 \in \left[-\frac{7}{4}, \frac{3}{2}\right]$ even though we haven't proved that the g we chose is any good (gives unique fixed point). I select $g(x) = (2x-1)^{1/3}$ (why? Because it seems the derivative $\frac{2}{3(2x-1)^{2/3}}$ can be potentially small and we are working with inputs between -1.5 and -1.75, definitely away from the vertical asymptote)

$$\begin{aligned}
p_1 &= g(p_0) = g(-1.6) = (2 \cdot (-1.6) - 1)^{1/3} \approx -1.613 \\
p_2 &= g(p_1) = g(-1.613) \approx -1.618 \\
p_3 &= g(p_2) = g(-1.618) \approx -1.618
\end{aligned}$$

Cool, it seems we are stabilizing to the golden ratio!

(e) (5 points) (Newton's Method) Use the same initial guess as in part (d) and relabel it as q_0. Use Newton's method on $f(x)$ and produce 3 iterates q_1, q_2, q_3.

Solution: Here we need the derivative of f, that is, $f'(x) = 3x^2 - 2$. Newton's method asks to iterate

$$\begin{aligned}
q_1 &= q_0 - \frac{f(q_0)}{f'(q_0)} = -1.6 - \frac{f(-1.6)}{f'(-1.6)} \approx -1.6183098 \\
q_2 &= q_1 - \frac{f(q_1)}{f'(q_1)} = -1.6183098 - \frac{f(-1.6183098)}{f'(-1.6183098)} \approx -1.6180341 \\
q_3 &= q_2 - \frac{f(q_1)}{f'(q_1)} = -1.618034 - \frac{f(-1.618034)}{f'(-1.618034)} \approx -1.6180340
\end{aligned}$$

I am including more digits here to see how good Newton's method is, in addition to a good initial guess narrowed down by the Bisection method.

(f) (5 points (bonus)) (Fixed-point Iteration) Prove that there exists a unique fixed-point on the interval for the $g(x)$ you proposed.

Looks like a good extra credit problem in your leisure time.

3. (30 points) Consider the system

$$
\begin{aligned}
3x_1 - x_2 + x_3 &= 1,\\
3x_1 + 6x_2 + 2x_3 &= 0,\\
3x_1 + 3x_2 + 7x_3 &= 4.
\end{aligned}
$$

(a) (15 points) Use the Jacobi method and produce 2 iterates, J_1, J_2 with $J_0 = (0,0,0)^{\mathrm{T}}$. You are welcome to use either formulation (matrix or element-wise).

Solution:

$$
\begin{aligned}
x_1^{(1)} &= \frac{b_1 - \left(a_{12}x_2^{(0)} + a_{13}x_3^{(0)}\right)}{a_{11}} = \frac{1-((-1)\cdot 0 + 1\cdot 0)}{3} = \frac{1}{3}\\
x_2^{(1)} &= \frac{b_2 - \left(a_{21}x_1^{(0)} + a_{23}x_3^{(0)}\right)}{a_{22}} = \frac{0-(3\cdot 0 + 2\cdot 0)}{6} = 0\\
x_3^{(1)} &= \frac{b_3 - \left(a_{31}x_1^{(0)} + a_{32}x_2^{(0)}\right)}{a_{33}} = \frac{4-(3\cdot 0 + 3\cdot 0)}{7} = \frac{4}{7}
\end{aligned}
$$

So, $J_1 = \left(\frac{1}{3}, 0, \frac{4}{7}\right)^{\mathrm{T}}$.

$$
\begin{aligned}
x_1^{(2)} &= \frac{b_1 - \left(a_{12}x_2^{(1)} + a_{13}x_3^{(1)}\right)}{a_{11}} = \frac{1-\left((-1)\cdot 0 + 1\cdot \frac{4}{7}\right)}{3} = \frac{1}{7}\\
x_2^{(2)} &= \frac{b_2 - \left(a_{21}x_1^{(1)} + a_{23}x_3^{(1)}\right)}{a_{22}} = \frac{0-\left(3\cdot \frac{1}{3} + 2\cdot \frac{4}{7}\right)}{6} = -\frac{5}{14}\\
x_3^{(2)} &= \frac{b_3 - \left(a_{31}x_1^{(1)} + a_{32}x_2^{(1)}\right)}{a_{33}} = \frac{4-\left(3\cdot \frac{1}{3} + 3\cdot 0\right)}{7} = \frac{3}{7}
\end{aligned}
$$

So, $J_2 = \left(\frac{1}{7}, -\frac{5}{14}, \frac{3}{7}\right)^{\mathrm{T}}$.

(b) (15 points) Use the Gauss-Seidel method and produce 2 iterates, G_1, G_2 with $G_0 = (0,0,0)^{\mathrm{T}}$. You are welcome to use either formulation (matrix or element-wise).

Solution:

$$
\begin{aligned}
x_1^{(1)} &= \frac{b_1 - \left(a_{12}x_2^{(0)} + a_{13}x_3^{(0)}\right)}{a_{11}} = \frac{1-((-1)\cdot 0 + 1\cdot 0)}{3} = \frac{1}{3}\\
x_2^{(1)} &= \frac{b_2 - \left(a_{21}\boxed{x_1^{(1)}} + a_{23}x_3^{(0)}\right)}{a_{22}} = \frac{0-\left(3\cdot \frac{1}{3} + 2\cdot 0\right)}{6} = -\frac{1}{6}\\
x_3^{(1)} &= \frac{b_3 - \left(a_{31}\boxed{x_1^{(1)}} + a_{32}\boxed{x_2^{(1)}}\right)}{a_{33}} = \frac{4-\left(3\cdot \frac{1}{3} + 3\cdot\left(-\frac{1}{6}\right)\right)}{7} = \frac{1}{2}
\end{aligned}
$$

So, $G_1 = \left(\frac{1}{3}, -\frac{1}{6}, \frac{1}{2}\right)$.

$$x_1^{(2)} = \frac{b_1 - \left(a_{12}x_2^{(1)} + a_{13}x_3^{(1)}\right)}{a_{11}} = \frac{1 - \left((-1)\cdot\left(-\frac{1}{6}\right) + 1\cdot\frac{1}{2}\right)}{3} = \frac{1}{9}$$

$$x_2^{(2)} = \frac{b_2 - \left(a_{21}\boxed{x_1^{(2)}} + a_{23}x_3^{(1)}\right)}{a_{22}} = \frac{0 - \left(3\cdot\frac{1}{9} + 2\cdot\frac{1}{2}\right)}{6} = -\frac{2}{9}$$

$$x_3^{(2)} = \frac{b_3 - \left(a_{31}\boxed{x_1^{(2)}} + a_{32}\boxed{x_2^{(2)}}\right)}{a_{33}} = \frac{4 - \left(3\cdot\frac{1}{9} + 3\cdot\left(-\frac{2}{9}\right)\right)}{7} = \frac{13}{21}$$

So, $G_2 = \left(\frac{1}{9}, -\frac{2}{9}, \frac{13}{21}\right)$. The boxed terms are the immediate updates from the previous line within the same step – a feature of the GS method.

4. (20 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \quad \lambda_1 = 5.$$

(a) (20 points) Use the power method to approximate the dominant eigenvalue up to an accuracy of 10^{-2} (you iterate until the error to the true value is less than this number). Use $\mathbf{x}^{(0)} = (1, 0)^{\mathrm{T}}$.

Solution:

$$\boldsymbol{x}^{(1)} = A\boldsymbol{x}^{(0)} = (1,4)^{\mathrm{T}}, \quad \lambda_1^{(1)} = \frac{\left\|\boldsymbol{x}^{(1)}\right\|}{\left\|\boldsymbol{x}^{(0)}\right\|} = \frac{4}{1} = 4.0000$$

$$\boldsymbol{x}^{(2)} = A\boldsymbol{x}^{(1)} = (9,16)^{\mathrm{T}}, \quad \lambda_2^{(1)} = \frac{\left\|\boldsymbol{x}^{(2)}\right\|}{\left\|\boldsymbol{x}^{(1)}\right\|} = \frac{16}{4} = 4.0000$$

$$\boldsymbol{x}^{(3)} = A\boldsymbol{x}^{(2)} = (41,84)^{\mathrm{T}}, \quad \lambda_3^{(1)} = \frac{\left\|\boldsymbol{x}^{(3)}\right\|}{\left\|\boldsymbol{x}^{(2)}\right\|} = \frac{84}{16} = 5.2500$$

$$\boldsymbol{x}^{(4)} = A\boldsymbol{x}^{(3)} = (209,416)^{\mathrm{T}}, \quad \lambda_4^{(1)} = \frac{\left\|\boldsymbol{x}^{(4)}\right\|}{\left\|\boldsymbol{x}^{(3)}\right\|} = \frac{416}{84} \approx 4.9524$$

$$\boldsymbol{x}^{(5)} = A\boldsymbol{x}^{(4)} = (1141,2084)^{\mathrm{T}}, \quad \lambda_5^{(1)} = \frac{\left\|\boldsymbol{x}^{(5)}\right\|}{\left\|\boldsymbol{x}^{(4)}\right\|} = \frac{2084}{416} \approx 5.0096$$

where $\left|\lambda_1^{(5)} - \lambda_1\right| < 10^{-2}$. Note that you can't use $\lambda_1^{(4)}$ because its error is still about $0.0476 > 10^{-2}$.

(b) (5 points (bonus)) What is the associated eigenvector for your last approximation of the eigenvalue? How can you check if this eigenvector is a good approximation without knowing the true eigenvector corresponding to $\lambda_1 = 5$?

Solution: The associated eigenvector is the normalized version of $\boldsymbol{x}^{(5)}$, namely,

$$\boldsymbol{v} = \frac{\boldsymbol{x}^{(5)}}{\left\|\boldsymbol{x}^{(5)}\right\|} = (0.54750, 1)^{\mathrm{T}}.$$

The eigenvalue-eigenvector relationship is $A\boldsymbol{v} = \lambda\boldsymbol{v}$. So we check if this equation is almost satisfied with $\lambda_1^{(5)}$ and $\boldsymbol{v}$. Indeed,

$$A\boldsymbol{v} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 0.54750 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.5475 \\ 5.1900 \end{bmatrix}$$

while

$$\lambda_1^{(5)}\boldsymbol{v} = 5.0096 \begin{bmatrix} 0.54750 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.7428 \\ 5.0096 \end{bmatrix}.$$

We have an error of

$$\left\|A\boldsymbol{v} - \lambda_1^{(5)}\boldsymbol{v}\right\|_\infty = 0.1953,$$

which is not bad after only 5 iterations.