MATH 3257 C01
Spring 2023 C Term
Exam 2
03/03/23
Time Limit: 50 Minutes

Name (Print): ______________________________

This exam contains 6 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books and notes. You may use a **scientific calculator** on this exam. You are allowed **one page (double-sided)** of notes.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- Manage the empty space wisely.
- Do NOT write in the table on the right.

Problem	Points	Score
1	20	
2	30	
3	30	
4	20	
Total:	100	

1. (20 points) True or False. Each problem has two components: the answer and the reason/correction, each worth 2 points. If true, state a concise reason or quote a theorem/homework. If false, give a counterexample or correct the statement by pointing out the logical fallacies (via some known results/theorems).

   (a) The Gauss-Seidel method is guaranteed to fail (to converge) when used to solve a system $Ax = b$ if A is not diagonally dominant.

   (b) A function $g(x)$ has a fixed point on the bounded interval $[a, b]$ if the function value satisfies $a \leq g(x) \leq b$ for every x that lies in $[a, b]$.

   (c) The Method of Successive Over-Relaxation (SOR) is an independent method, completely different from Gauss-Seidel and Jacobi iteration.

   (d) The Power Method for estimating the dominant eigenvalue of a matrix converges very fast if the largest and the second largest eigenvalue are very close in magnitude.

   (e) Applying the Secant Method to solve the root-finding problem $f(x) = 0$ needs the derivative of f.

2. (30 points) Consider the root-finding problem

$$f(x) = x^3 - 2x + 1 = 0.$$

(a) (10 points) Use the Intermediate Value Theorem to provide a reasonable interval where a root is guaranteed to exist.

(b) (5 points) (Bisection) Use the Bisection method on this interval, and produce 3 iterates, b_1, b_2, b_3. Evaluate $f(b_i)$ for $i = 1, 2, 3$. Is the value getting closer to zero?

(c) (5 points) (Fixed-point Iteration) Propose a function $g(x)$ such that its fixed-point satisfies $f(x) = 0$.

(d) (5 points) (Fixed-point Iteration) Pick any initial guess p_0 that lies in the interval you found from part (a). Iterate $p_n = g\left(p_{n-1}\right)$ 3 times, i.e., obtain p_1, p_2, p_3.

(e) (5 points) (Newton's Method) Use the same initial guess as in part (d) and relabel it as q_0. Use Newton's method on $f\left(x\right)$ and produce 3 iterates q_1, q_2, q_3.

(f) (5 points (bonus)) (Fixed-point Iteration) Prove that there exists a unique fixed-point on the interval for the $g\left(x\right)$ you proposed.

3. (30 points) Consider the system

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 1, \\ 3x_1 + 6x_2 + 2x_3 &= 0, \\ 3x_1 + 3x_2 + 7x_3 &= 4. \end{aligned}$$

(a) (15 points) Use the Jacobi method and produce 2 iterates, J_1, J_2 with $J_0 = (0,0,0)^{\mathrm{T}}$. You are welcome to use either formulation (matrix or element-wise).

(b) (15 points) Use the Gauss-Seidel method and produce 2 iterates, G_1, G_2 with $G_0 = (0,0,0)^{\mathrm{T}}$. You are welcome to use either formulation (matrix or element-wise).

4. (20 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \quad \lambda_1 = 5.$$

(a) (20 points) Use the power method to approximate the dominant eigenvalue up to an accuracy of 10^{-2} (you iterate until the error to the true value is less than this number). Use $\mathbf{x}^{(0)} = (1, 0)^{\mathrm{T}}$.

(b) (5 points (bonus)) What is the associated eigenvector for your last approximation of the eigenvalue? How can you check if this eigenvector is a good approximation without knowing the true eigenvector corresponding to $\lambda_1 = 5$?