MATH 3257 C01 **Name (Print):** SOLUTION KEY
Spring 2023 C Term
Exam 1
02/10/23
Time Limit: 50 Minutes

This exam contains 8 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books and notes. You may use a **scientific calculator** on this exam. You are allowed **one page (single-sided)** of notes.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- Manage the empty space wisely.
- Do NOT write in the table on the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) True or False. Each problem has two components: the answer and the reason/correction, each worth 2 points. If true, state a concise reason or quote a theorem/homework. If false, give a counterexample or correct the statement.

   (a) 0.03140×10^3 has 6 significant digits.

   Solution: False. This number has 4 significant digits.

   (b) If two real numbers can be represented **exactly** as floating-point numbers, then the result of a real arithmetic operation on them can also be represented exactly as a floating-point number.

   Solution: False. Both 1 and 3 can be represented exactly by floating-point numbers: $1 = (-1)^0 2^0$ and $3 = (-1)^0 2^1 \left(1 + \frac{1}{2}\right)$. But $\frac{1}{3}$ certainly can't be.

   (c) A square matrix with real entries always has real eigenvalues.

   Solution: False. Both matrices in HW4 Q5 have real entries but complex eigenvalues.

   (d) k-digit chopping of a real number incurs at least as much round-off error as k-digit rounding.

   Solution: True. HW1 Q5.

   (e) One of the goals of Gaussian elimination with partial pivoting is to reduce round-off errors.

   Solution: True. Round-off error is mitigated by swapping the rows so that the multipliers are formed by dividing a pivot with larger magnitude.

2. (20 points) We are interested in a stable algorithm to evaluate the function

$$f(x) = \sqrt{1+x} - \sqrt{1-x}, \quad x \approx 0.$$

(a) (10 points) Find κ_{rel}, the relative condition number of this function near $x = 0$. Is the function well-conditioned near $x = 0$ (say $\kappa_{\text{rel}} \leq 2$)?

Solution:

$$\kappa_{\text{rel}}(x) = \left| \frac{x f'(x)}{f(x)} \right|$$

where we need to find the derivative of the function.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[(1+x)^{1/2} - (1-x)^{1/2}\right] \\ &= \frac{1}{2}(1+x)^{-1/2} - \frac{1}{2}(1-x)^{-1/2}(-1) \\ &= \frac{1}{2}\left[\frac{1}{\sqrt{1+x}} + \frac{1}{\sqrt{1-x}}\right] \end{aligned}$$

Thus, by rationalizing $\sqrt{1+x} - \sqrt{1-x}$ with its conjugate (at step **A** below), we have

$$\begin{aligned} \kappa_{\text{rel}}(x) &= \left| \frac{x f'(x)}{f(x)} \right| \\ &= \left| \frac{\frac{x}{2}\left[\frac{1}{\sqrt{1+x}} + \frac{1}{\sqrt{1-x}}\right]}{\sqrt{1+x} - \sqrt{1-x}} \right| \\ &= \frac{1}{2} \frac{x}{\sqrt{1+x} - \sqrt{1-x}} \left(\frac{1}{\sqrt{1+x}} + \frac{1}{\sqrt{1-x}}\right) \\ &\overset{\mathbf{A}}{=} \frac{1}{2} \frac{x\left(\sqrt{1+x} + \sqrt{1-x}\right)}{\left(\sqrt{1+x} - \sqrt{1-x}\right)\left(\sqrt{1+x} + \sqrt{1-x}\right)} \left(\frac{1}{\sqrt{1+x}} + \frac{1}{\sqrt{1-x}}\right) \\ &= \frac{1}{2} \frac{x\left(\sqrt{1+x} + \sqrt{1-x}\right)}{2x} \left(\frac{1}{\sqrt{1+x}} + \frac{1}{\sqrt{1-x}}\right) \\ &= \frac{\sqrt{1+x} + \sqrt{1-x}}{4} \left(\frac{1}{\sqrt{1+x}} + \frac{1}{\sqrt{1-x}}\right) \\ &\to 1, \quad x \to 0 \end{aligned}$$

which shows that the function is well-conditioned near $x = 0$.

Grading rubric:

- Definition of relative condition number for a function: 5 points.
- Correct derivative: 2 points.
- Knowing to rationalize: 2 points.
- Correct conclusion: 1 point.

(b) (4 points) List the machine operations for a **naive** (direct) evaluation algorithm.

Solution:

1. + and - (concurrently, though not necessary), labeled as s_1^+ and s_1^-.
2. square roots (concurrently, though not necessary), labeled as $s_2^\pm$
3. subtraction, labeled as s_3.

(c) (3 points) Determine the stability of the algorithm in part(b) by examining the stability of each step.

Solution:

$$s_1^+(x) = 1 + x, \quad \kappa_{s_1}^+(x) = \left| \frac{x \frac{ds_1^+}{dx}}{s_1^+} \right| = \left| \frac{x \cdot (1)}{1 + x} \right| \to 0, \quad \text{as } x \to 0;$$

$$s_1^-(x) = 1 - x, \quad \kappa_{s_1}^-(x) = \left| \frac{x \frac{ds_1^-}{dx}}{s_1^-} \right| = \left| \frac{x \cdot (-1)}{1 - x} \right| \to 0, \quad \text{as } x \to 0;$$

$$s_2^\pm := s_2\left(s_1^\pm\right) = \sqrt{s_1^\pm}, \quad \kappa_{s_2}(x) = \left| \frac{s_1^\pm \frac{ds_2}{ds_1^\pm}}{s_2} \right| = \left| \frac{s_1^\pm \cdot \frac{\pm 1}{2\sqrt{s_1^\pm}}}{\sqrt{s_1^\pm}} \right| \to \frac{1}{2}, \quad \text{as } s_1^\pm \to 1.$$

$$s_3\left(s_2^+, s_2^-\right) = s_2^+ - s_2^-, \quad \kappa_{s_3;s_2^+}\left(s_2^+, s_2^-\right) = \left| \frac{s_2^+ \frac{\partial s_3\left(s_2^+, s_2^-\right)}{\partial s_2^+}}{s_3} \right| = \left| \frac{s_2^+ \cdot (1)}{s_2^+ - s_2^-} \right|,$$

which blows up as $\left(s_2^+, s_2^-\right) \to (1,1)$. There is then no need to check $\kappa_{s_3;s_2^-}$ (though it leads to the same conclusion).

(d) (3 points) Propose a stable algorithm to evaluate this function.

Solution:

$$\begin{aligned} f(x) &= \sqrt{1+x} - \sqrt{1-x}. \\ &= \frac{\left(\sqrt{1+x} - \sqrt{1-x}\right)\left(\sqrt{1+x} + \sqrt{1-x}\right)}{\sqrt{1+x} + \sqrt{1-x}} \\ &= \frac{1 + x - (1-x)}{\sqrt{1+x} + \sqrt{1-x}} \\ &= \frac{2x}{\sqrt{1+x} + \sqrt{1-x}} \end{aligned}$$

avoids catastrophic cancellation by circumventing the subtraction.

(e) (5 points (bonus)) Prove that the algorithm in part(d) is stable. You may reuse some of the results from previous steps.

Solution: A nice extra credit problem for HW.

3. (20 points) Use **two-digit rounding** arithmetic to solve the following system of equations

$$\begin{aligned} 4x_1 + 2x_2 - x_3 &= -5, \\ \frac{1}{9}x_1 + \frac{1}{9}x_2 - \frac{1}{3}x_3 &= -1, \\ x_1 + 4x_2 + 2x_3 &= 9. \end{aligned}$$

Solution: In augmented form, we have

$$\left[\begin{array}{ccc|c} 4 & 2 & -1 & -5 \\ \frac{1}{9} & \frac{1}{9} & -\frac{1}{3} & -1 \\ 1 & 4 & 2 & 9 \end{array}\right].$$

With 2-digit rounding, we start with

$$\left[\begin{array}{ccc|c} 4 & 2 & -1 & -5 \\ 0.11 & 0.11 & -0.33 & -1 \\ 1 & 4 & 2 & 9 \end{array}\right]$$

and proceed with Gaussian elimination (with or without pivoting, your choice). First, we see the multiplier using the first row is

$$m_{21} = \frac{0.11}{4} = 0.028, \quad m_{31} = \frac{1}{4} = 0.25.$$

Then,

$$\left[\begin{array}{ccc|c} 4 & 2 & -1 & -5 \\ 0.11 & 0.11 & -0.33 & -1 \\ 1 & 4 & 2 & 9 \end{array}\right] \overset{(E_2 - 0.028E_1)\to E_2}{\to} \left[\begin{array}{ccc|c} 4 & 2 & -1 & -5 \\ 0 & 0.054 & -0.30 & -0.86 \\ 1 & 4 & 2 & 9 \end{array}\right]$$

$$\overset{(E_3 - 0.25E_1)\to E_3}{\to} \left[\begin{array}{ccc|c} 4 & 2 & -1 & -5 \\ 0 & 0.054 & -0.30 & -0.86 \\ 0 & 3.5 & 2.3 & 10 \end{array}\right]$$

$$\overset{(E_3 - 65E_2)\to E_3}{\to} \left[\begin{array}{ccc|c} 4 & 2 & -1 & -5 \\ 0 & 0.054 & -0.30 & -0.86 \\ 0 & 0 & 22 & 66 \end{array}\right]$$

where we used the second row multiplier $m_{32} = \frac{3.5}{0.054} = 65$. Thus, with back substitution, we obtain

$$\begin{aligned} x_3 &= \frac{66}{22} = 3.0, \\ x_2 &= \frac{-0.86 - (-0.30)(3.0)}{0.054} = \frac{-0.86 + 0.90}{0.054} = \frac{0.040}{0.054} = 0.74, \\ x_1 &= \frac{-5 - (-1)(3.0) - (2)(0.74)}{4} = \frac{-5 + 3.0 - 1.5}{4} = \frac{-3.5}{4} = -0.88. \end{aligned}$$

If one partially pivots at the second row into

$$\left[\begin{array}{ccc|c} 4 & 2 & -1 & -5 \\ 0 & 0.054 & -0.30 & -0.86 \\ 0 & 3.5 & 2.3 & 10 \end{array}\right] \overset{E_2 \leftrightarrow E_3}{\rightarrow} \left[\begin{array}{ccc|c} 4 & 2 & -1 & -5 \\ 0 & 3.5 & 2.3 & 10 \\ 0 & 0.054 & -0.30 & -0.86 \end{array}\right]$$

and then reduce using $m_{32} = \frac{0.054}{3.5} = 0.015$, we have

$$\left[\begin{array}{ccc|c} 4 & 2 & -1 & -5 \\ 0 & 3.5 & 2.3 & 10 \\ 0 & 0.054 & -0.30 & -0.86 \end{array}\right] \overset{(E_3 - 0.015E_2) \rightarrow (E_3)}{\rightarrow} \left[\begin{array}{ccc|c} 4 & 2 & -1 & -5 \\ 0 & 3.5 & 2.3 & 10 \\ 0 & 0 & -0.34 & -1.0 \end{array}\right].$$

Thus, with back substitution, we have

$$\begin{aligned} x_3 &= \frac{-1.0}{-0.34} = 2.9 \\ x_2 &= \frac{10 - 2.3 \cdot 2.9}{3.5} = \frac{10 - 6.7}{3.5} = 0.94 \\ x_1 &= \frac{-5 - (-1)\,(2.9) - 2 \cdot (0.94)}{4} = \frac{-5 + 2.9 - 1.9}{4} = -1 \end{aligned}$$

Note that the true solution is $\mathbf{x} = (-1, 1, 3)$, where partial pivoting $\mathbf{x}_{\text{pv}} = (-1, 0.94, 2.9)$ does yield a better solution than naive GE $\mathbf{x}_{\text{nge}} = (-0.88, 0.74, 3.0)$. More precisely, we compute the l^∞-error,

$$\begin{aligned} \|\boldsymbol{x}_{\text{pv}} - \boldsymbol{x}\|_\infty &= \left\|(-1, 0.94, 2.9)^{\mathrm{T}} - (-1, 1, 3)^{\mathrm{T}}\right\|_\infty = \left\|(0, -0.06, -0.1)^{\mathrm{T}}\right\|_\infty = 0.1 \\ \|\boldsymbol{x}_{\text{nge}} - \boldsymbol{x}\|_\infty &= \left\|(-0.88, 0.74, 3.0)^{\mathrm{T}} - (-1, 1, 3)^{\mathrm{T}}\right\|_\infty = \left\|(0.12, -0.26, 0)^{\mathrm{T}}\right\|_\infty = 0.26. \end{aligned}$$

Grading rubric:

- Augmented form: 2 points. If you don't have this but a clean presentation/organization, then 2 points as well. Otherwise, 0 point.
- Multipliers: 3 points in total, 1 for each.
- Elementary row operations: 9 points in total, 3 for each.
- Back substitutions: 6 points in total, 2 for each.

Partial credits are given according to the rubric based on your understanding of the rounding concept and not particularly on the correctness of the numbers in the case where mistakes have been made.

4. (20 points) We've learned from class that for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have the Cauchy-Schwarz inequality

$$\mathbf{x} \cdot \mathbf{y} \leq \|\mathbf{x}\|_2 \|\mathbf{y}\|_2.$$

where $\cdot$ is the dot product (between vectors of equal dimension). Verify that the inequality holds for the following pairs of vectors.

(a) (10 points) $\mathbf{x} = (2, 3, -2)^{\mathrm{T}}$ and $\mathbf{y} = (-1, 5, 6)^{\mathrm{T}}$.
Solution:

$$LHS = \mathbf{x} \cdot \mathbf{y} = (2, 3, -2) \cdot (-1, 5, 6) = (2)(-1) + (3)(5) + (-2)(6) = -2 + 15 - 12 = 1,$$

while

$$RHS = \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 = \sqrt{2^2 + 3^2 + (-2)^2} \sqrt{(-1)^2 + 5^2 + 6^2} = \sqrt{17}\sqrt{62}.$$

Clearly, $1 \leq \sqrt{17}\sqrt{62}$, so the inequality holds.

(b) (10 points) $\mathbf{x} = \left(e^2, \pi^{-3.5}, 0\right)^{\mathrm{T}}$ and $\mathbf{y} = \left(-e^{-2}, \pi^{3.5}, 101\right)^{\mathrm{T}}$.
Solution:

$$LHS = \mathbf{x} \cdot \mathbf{y} = \left(e^2, \pi^{-3.5}, 0\right) \cdot \left(-e^{-2}, \pi^{3.5}, 101\right) = -1 + 1 + 0 = 0,$$

while we observe the RHS is the product of two l^2 norms, which must be nonnegative. The inequality holds.

(c) (5 points (bonus)) For what $\mathbf{x}$ and $\mathbf{y}$ does Cauchy-Schwarz inequality achieve equality?
Solution: By the dot product definition,

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \cos\theta$$

where θ is the angle between the two vectors. Thus, equality is achieved when $\cos\theta = 1$, or $\theta = 0$, namely, when the two vectors are in the same direction. In other words, $\mathbf{x} = k\mathbf{y}$ for some $k \in \mathbb{R}$ (including the case when one of them is the 0 vector).

5. (20 points) Find the eigenvalues and the associated eigenvectors of the following matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Solution:

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = 0$$

which provides us with the characteristic polynomial

$$(1-\lambda)^2 - 4 = 0 \implies 1 - \lambda = \pm 2 \implies \lambda_1 = -1, \quad \lambda_2 = 3.$$

We then find the eigenvector corresponding to each eigenvalue.

- $\lambda_1 = -1$
  We solve
  $$\begin{bmatrix} 1-\lambda_1 & 2 \\ 2 & 1-\lambda_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \implies \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0 \implies \mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

- $\lambda_2 = 3$
  We solve
  $$\begin{bmatrix} 1-\lambda_2 & 2 \\ 2 & 1-\lambda_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0 \implies \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \implies \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Grading rubric:

- $A - \lambda I$ form: 3 points.
- Characteristic polynomial: 5 points.
- Eigenvalues: 4 points, 2 each.
- Eigenvectors: 8 points, 4 each.