MATH 3257 C01
Spring 2023 C Term
Exam 1
02/10/23
Time Limit: 50 Minutes

Name (Print): ______________________________

This exam contains 7 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books and notes. You may use a **scientific calculator** on this exam. You are allowed **one page (single-sided)** of notes.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- Manage the empty space wisely.
- Do NOT write in the table on the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) True or False. Each problem has two components: the answer and the reason/correction, each worth 2 points. If true, state a concise reason or quote a theorem/homework. If false, give a counterexample or correct the statement.

   (a) 0.03140×10^3 has 6 significant digits.

   (b) If two real numbers can be represented **exactly** as floating-point numbers, then the result of a real arithmetic operation on them can also be represented exactly as a floating-point number.

   (c) A square matrix with real entries always has real eigenvalues.

   (d) k-digit chopping of a real number incurs at least as much round-off error as k-digit rounding.

   (e) One of the goals of Gaussian elimination with partial pivoting is to reduce round-off errors.

2. (20 points) We are interested in a stable algorithm to evaluate the function

$$f(x) = \sqrt{1+x} - \sqrt{1-x}, \quad x \approx 0.$$

(a) (10 points) Find κ_{rel}, the relative condition number of this function near $x = 0$. Is the function well-conditioned near $x = 0$ (say $\kappa_{\text{rel}} \leq 2$)?

(b) (4 points) List the machine operations for a **naive** (direct) evaluation algorithm.

(c) (3 points) Determine the stability of the algorithm in part(b) by examining the stability of each step.

(d) (2 points) Propose a stable algorithm to evaluate this function.

(e) (5 points (bonus)) Prove that the algorithm in part(d) is stable. You may reuse some of the results from previous steps.

3. (20 points) Use **two-digit rounding** arithmetic to solve the following system of equations

$$
\begin{aligned}
4x_1 + 2x_2 - x_3 &= -5,\\
\frac{1}{9}x_1 + \frac{1}{9}x_2 - \frac{1}{3}x_3 &= -1,\\
x_1 + 4x_2 + 2x_3 &= 9.
\end{aligned}
$$

4. (20 points) We've learned from class that for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have the Cauchy-Schwarz inequality

$$\mathbf{x} \cdot \mathbf{y} \leq \|\mathbf{x}\|_2 \|\mathbf{y}\|_2.$$

where $\cdot$ is the dot product (between vectors of equal dimension). Verify that the inequality holds for the following pairs of vectors.

(a) (10 points) $\mathbf{x} = (2, 3, -2)^{\mathrm{T}}$ and $\mathbf{y} = (-1, 5, 6)^{\mathrm{T}}$.

(b) (10 points) $\mathbf{x} = \left(e^2, \pi^{-3.5}, 0\right)^{\mathrm{T}}$ and $\mathbf{y} = \left(-e^{-2}, \pi^{3.5}, 101\right)^{\mathrm{T}}$.

(c) (5 points (bonus)) For what $\mathbf{x}$ and $\mathbf{y}$ does Cauchy-Schwarz inequality achieve equality?

5. (20 points) Find the eigenvalues and the associated eigenvectors of the following matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$