MATH 3257 C03
Spring 2026 C Term
Exam 2
03/06/2025
Time Limit: 50 Minutes

Name (Print): ______________________________

This exam contains 7 pages (including this cover page) and 4 problems (including 1 bonus). Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books and lecture notes. You may use a non-programmable scientific calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- Define new variables/symbols when you introduce them.
- Manage the empty space wisely.
- Do NOT write in the table on the right.

Problem	Points	Score
1	15	
2	15	
3	15	
4	0	
Total:	45	

1. (15 points) **(Very) Short** Responses.

   (a) (3 points) In Power Iteration for the spectral radius, why do we need to rescale the iterates $Ax^{(k)}$?

   (b) (3 points) Suppose $g : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and has a unique fixed point $p = g(p)$. What property of g at the fixed point controls the rate of convergence of the fixed-point iteration $x^{(k+1)} = g(x^{(k)})$?

   (c) (3 points) Name one practical advantage of the Secant Method over the Newton-Raphson method for root-finding problems.

   (d) (3 points) Let $G(x_1, x_2) = \begin{pmatrix} g_1(x_1, x_2) \\ g_2(x_1, x_2) \end{pmatrix}$. Explain the difference between a naive fixed-point iteration and its Gauss-Seidel variant by writing out the update rule for both, and boxing the key difference.

   (e) (3 points) Consider minimizing $\phi(x) = \frac{1}{2}x^2$ by gradient descent $x^{(k+1)} = x^{(k)} - \alpha\phi'(x^{(k)})$. Derive the iteration in the form $x^{(k+1)} = c(\alpha)x^{(k)}$, and give a condition on α for convergence, and an explicit choice of α that diverges.

2. (15 points) Consider the nonlinear equation

$$f(x) = x^3 - 8 = 0.$$

We convert this root-finding problem into a fixed-point iteration by defining, for a parameter $\alpha \neq 0$,

$$g(x) = x - \alpha f(x), \qquad x^{(k+1)} = g\left(x^{(k)}\right).$$

Let p denote the root.

(a) (4 points) Show that $f(p) = 0$ if and only if $p = g(p)$. (In other words, justify that solving $f(x) = 0$ is equivalent to finding a fixed point of g.)

(b) (4 points) Confirm that $p = 2$ is a root of f. Show that any $\alpha \in (0, 1/6)$ guarantees local convergence of the iteration to $p = 2$ (should the initial guess be close enough).

(c) (4 points) Derive the Newton-Raphson iteration formula for solving $f(x) = 0$. Briefly justify why Newton's method is expected to converge *quadratically* near $p = 2$.

(d) (3 points) Take $x^{(0)} = 1.5$.

- Choose a value of α in the interval shown in part (b), and compute two iterates $x^{(1)}, x^{(2)}$ of the fixed-point iteration.
- Compute two iterates $x^{(1)}, x^{(2)}$ of Newton's method.

Based on these two iterates, which method appears to converge faster toward $p = 2$? Is your result surprising? Why or why not?

	Fixed-point iteration	**Newton's method**
$x^{(1)}$		
$x^{(2)}$		

Discussion:

3. (15 points) Consider the minimization problem $\phi(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$ for $x \in [-2, 2]$.

   (a) (4 points) Using standard calculus techniques (critical points + second derivative test), show that the function $\phi(x)$ has global minima at $x = \pm 1$ on the interval $[-2, 2]$.

   (b) (4 points) Construct the naive Gradient Descent iteration with constant step size $\alpha > 0$.

   (c) (4 points) Treating gradient descent as a fixed-point iteration $x^{(k+1)} = g(x^{(k)})$, verify that $x = -1, 0, 1$ are fixed points of the iteration.

(d) (3 points) (Fill in the blank first) Recall that a fixed-point iteration $x^{(k+1)} = g(x^{(k)})$ converges locally to a fixed point x^* if

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Apply this criterion to each fixed point found in part (c). Determine the condition on the step size α under which the iteration converges near each fixed point.

(e) (3 points (bonus)) Sketch $\phi(x)$ and explain why the stability results from part (d) are consistent with the geometry of the function.

4. (2 points (bonus)) Pick your most memorable topic learned in this course, and introduce it to me. Show some math and some discussion on why it is memorable/interesting to you, and possibly, to your major.