MATH 3257 C03
Spring 2026 C Term
Exam 1
02/12/2025
Time Limit: 50 Minutes

Name (Print): ____________________

This exam contains 7 pages (including this cover page) and 3 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books and lecture notes.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- Leave your answers in exact forms, such as $\sqrt{2}$ or $1/5$. Do not use decimal representations of numbers **unless instructed to do so**.
- Define new variables/symbols when you introduce them.
- Manage the empty space wisely.
- Do NOT write in the table on the right.

Problem	Points	Score
1	15	
2	15	
3	15	
Total:	45	

1. (15 points) **(Very) Short** Responses.

   (a) (3 points) In Gaussian elimination, rows are swapped during *partial pivoting*. Explain what partial pivoting accomplishes and **how** it is done.

   (b) (3 points) Suppose the matrix A admits an LU factorization $A = LU$, where L is lower triangular and U is upper triangular. Describe **how** to efficiently solve the linear system $Ax = b$ using this factorization.

   (c) (3 points) Name one topic from this course in which the spectral radius of a matrix plays a central role. Briefly explain why the spectral radius is important in that context.

   (d) (3 points) A student applies the Jacobi iteration to solve the linear system $Ax = b$ and observes that the method diverges after 20 iterations. List *two distinct* reasons why the Jacobi method **may** fail to converge for a given system.

   (e) (3 points) In lecture, we learned both the component form and the matrix form of Jacobi, Gauss-Seidel and Successive Over-relaxation. **Why** do we need the matrix form, even though we can already implement the methods numerically using the component form?

2. (15 points) Consider the system

$$\begin{aligned} 2x_1 - x_2 &= 1, \\ -x_1 + 2x_2 &= 1. \end{aligned}$$

We solve it using the Jacobi iteration and the Gauss-Seidel iteration. The true solution is $(1, 1)$.

(a) (3 points) Do you expect the Jacobi iteration to converge and why? No need to prove anything, but provide a mathematical observation to **justify** your answer.

(b) (5 points) Use the Jacobi method and produce 2 iterates, J_1, J_2 (both are vectors) with $J_0 = (0, 0)^{\mathrm{T}}$. You are welcome to use either formulation (matrix or component).

$J_1 =$ $\boxed{}$

$J_2 =$ $\boxed{}$

(c) (5 points) Use the Gauss-Seidel method and produce 2 iterates, G_1, G_2 (both are vectors) with $G_0 = (0, 0)^{\mathrm{T}}$. You are welcome to use either formulation (matrix or component).

$G_1 =$ $\boxed{}$

$G_2 =$ $\boxed{}$

(d) (2 points) How would you **set up** a comparison of the rate of convergence between the two methods? Answer in both the theoretical and numerical aspect.

3. (15 points) **Remark.** Parts of this problem build on earlier parts. If you are unable to complete one part, you may assume its result (correctly stated) and continue with the remaining parts. You will receive full credit for correct work that follows logically from your stated assumption.

   Consider a linear iterative method for solving $A\mathbf{x} = \mathbf{b}$ written in fixed-point form

   $$\mathbf{x}^{(k+1)} = T\mathbf{x}^{(k)} + \mathbf{c}.$$

   Assume the exact solution $\mathbf{x}^*$ satisfies $A\mathbf{x}^* = \mathbf{b}$.

   (a) (5 points) **Error recursion.**
   Define the error

   $$\mathbf{e}^{(k)} := \mathbf{x}^* - \mathbf{x}^{(k)}.$$

   (i) Show that $\mathbf{x}^*$ satisfies $\mathbf{x}^* = T\mathbf{x}^* + \mathbf{c}$.

   (ii) Derive the error recursion

   $$\mathbf{e}^{(k+1)} = T\mathbf{e}^{(k)}.$$

   (b) (5 points) **Residual recursion.**
   Define the residual

   $$\mathbf{r}^{(k)} := \mathbf{b} - A\mathbf{x}^{(k)}.$$

   (i) Show that

   $$\mathbf{r}^{(k)} = A\mathbf{e}^{(k)}.$$

   (ii) Using part (a), show that

   $$\mathbf{r}^{(k+1)} = ATA^{-1}\mathbf{r}^{(k)}.$$

(c) (5 points) **Richardson (residual) form.**
Suppose the iteration is written instead in the form

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + B^{-1}(\mathbf{b} - A\mathbf{x}^{(k)}),$$

where B is an invertible matrix.

(i) Show that this iteration can be written in fixed-point form with

$$T = I - B^{-1}A.$$

(ii) Using your work in part (b) (last page), show that

$$\mathbf{r}^{(k+1)} = (I - AB^{-1})\mathbf{r}^{(k)}.$$

(d) (5 points (bonus)) **Similarity and convergence insight.**
Recall that two matrices M and N are called *similar* if there exists an invertible matrix S such that

$$N = S^{-1}MS.$$

(i) Show that the matrices

$$T = I - B^{-1}A \qquad \text{and} \qquad I - AB^{-1}$$

are similar. [Hint: choose $S = A$.]

(ii) Show that similar matrices have the same eigenvalues. [Hint: the characteristic polynomial of a matrix M is $p_M(\lambda) = \det(\lambda I - M)$.]

(iii) Why is this fact important when studying convergence of iterative methods?