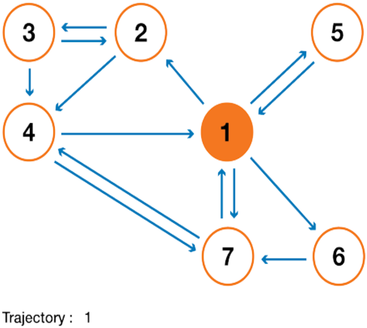
Introduction to Markov Chains

Open Your Eyes



October 9th 2025

From Deterministic to Probabilistic Thinking (I)

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- The system is too complex (weather, finance, biology).
- We do not observe all relevant variables.
- Measurement errors or noise accumulate.

Result: Our best description is not a trajectory, but a *distribution of possible states*.

From Deterministic to Probabilistic Thinking (II)

A **Markov chain** describes dynamics through *probabilities*, not precise values.

- Instead of $x(t)$, we track the probability that the system, as a random variable X_t, is in **each possible state**, using a vector

$$\boldsymbol{\pi}^{(t)} = \left[P\left(X_t = 1\right), P\left(X_t = 2\right), \ldots, P\left(X_t = n\right)\right]$$

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In short: deterministic dynamics describe *what must happen*; Markov dynamics describe *what tends to happen*.

What is a Markov Chain?

A **Discrete State Discrete Time Markov chain** is a sequence of random variables

$$X_0, X_1, X_2, \ldots, \quad X_i \in \text{Countable or Finite Set}$$

that satisfies the **Markov property**:

$$P(X_{n+1} = x_{n+1} \mid X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) \\ = P(X_{n+1} = x_{n+1} \mid X_n = x_n).$$

The future depends only on the present, not on the past. In other words, the future state forgets about history – **memoryless**.

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- Number of customers in the queue and at the server.

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- Weather transitions: sunny $\leftrightarrow$ cloudy $\leftrightarrow$ rainy.
- Number of customers in the queue and at the server.
- Machine states: operational $\leftrightarrow$ failed.

Transition Probability Matrix

The transitions between states are summarized in a matrix:

$$P = [p_{ij}], \qquad p_{ij} = P(X_{n+1} = j \mid X_n = i),$$

where each row represents the probabilities of leaving state i.

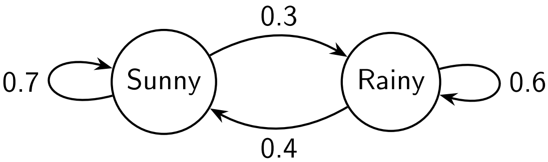
Properties

- $p_{ij} \geq 0$ for all i, j, (Axiom 1)
- $\sum_j p_{ij} = 1$ for each i. (Axiom 3)

Example: A Simple Weather Model

Example: Two-state weather model

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}, \quad \text{States: } 1 = \text{Sunny},\ 2 = \text{Rainy}.$$



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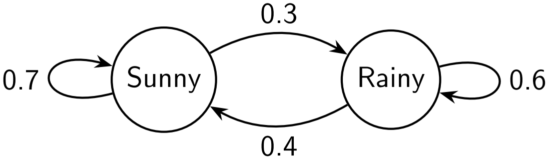

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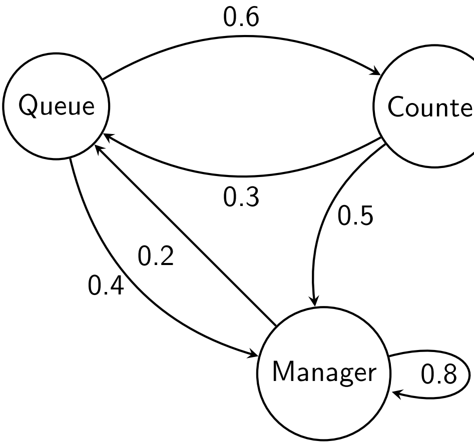
Real weather models are far more complex, but they do involve basic steps that use (continuous-time) Markov chains.

Example: Three-State System



$$P = \begin{bmatrix} 0 & 0.6 & 0.4 \\ 0.3 & 0 & 0.7 \\ 0.2 & 0 & 0.8 \end{bmatrix}$$

Interpretation:

- Rows sum to 1 (law of total probability).
- $P_{ij} = \text{Prob}(\text{next state } j \mid \text{current state } i)$.
- The matrix drives how probabilities evolve in time.

n-step Transitions

The n-step transition probabilities are defined by

$$p_{ij}^{(n)} = P(X_{n+k} = j \mid X_k = i).$$

They are obtained from powers of the transition matrix:

$$P^{(n)} = P^n. \text{ Why? Next slide.}$$

In words, the n-step transition matrix is the one-step matrix raised to the pth power.

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In words, the n-step transition matrix is the one-step matrix raised to the pth power. **Example:** If

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}, \quad \text{then} \quad P^2 = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix}.$$

After two days, the chain “forgets” a bit of its starting point.

The Multiplication Rule Revisited

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Interpretation:

- We sum over all possible intermediate events $B = b$.
- The second law expresses how we “go through” the middle layer B.

Why Does $\mathbf{P}^{(2)} = \mathbf{P}^2$?

Idea: A two-step transition probability is just a *sum over all possible intermediate states*, by the **law of total (conditional) probabilities.**

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Interpretation:

- P_{ik} = probability of going from i to k in one step.
- P_{kj} = probability of going from k to j in one step.
- Summing over k accounts for *all possible middle stops.*

Why (and how) Does the Transition Matrix Evolve the Distribution?

Let the system have states $1, 2, \ldots, m$. At time t, the distribution is

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Interpretation:

- The transition matrix does not change individual outcomes.
- It *moves probability mass* across states.
- This is how the distribution *evolves in time*.

Setup (states)

We consider a discrete state space $\Omega = \{0, 1, 2, 3\}$ (positions along a short line/cycle). Start the system at state 0 (initial distribution $\pi^{(0)} = (1, 0, 0, 0)$).

We compare:

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- A deterministic *permutation* dynamics (constant-speed motion around the 4 states).
- A realistic-looking *Markov* dynamics with probabilistic transitions (local moves, some inertia).

Transition matrices

Deterministic (permutation) matrix:

$$P_{\mathrm{det}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

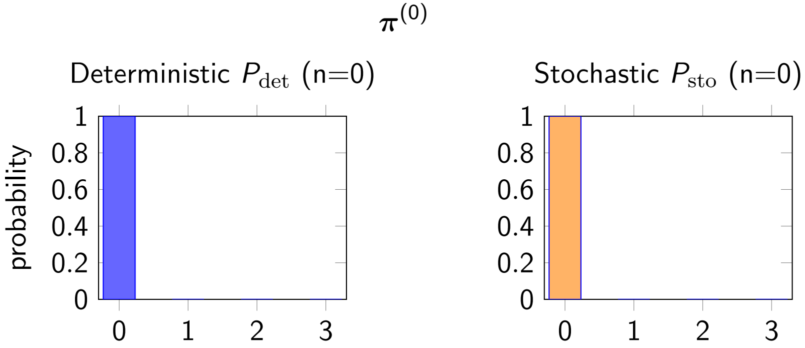
This maps $0 \mapsto 1 \mapsto 2 \mapsto 3 \mapsto 0$: a perfect clockwise step each tick.

Stochastic Markov matrix (richer behaviour):

$$P_{\mathrm{sto}} = \begin{pmatrix} 0.10 & 0.90 & 0 & 0 \\ 0.20 & 0.10 & 0.70 & 0 \\ 0 & 0.30 & 0.10 & 0.60 \\ 0 & 0 & 0.40 & 0.60 \end{pmatrix}$$

Rows sum to 1; entries $p_{ij} = P(\text{next} = j \mid \text{current} = i)$.

Time evolution: $n = 0$



Initial distribution $\pi^{(0)} = (1, 0, 0, 0)$ for both models.

Time evolution: $n = 1$

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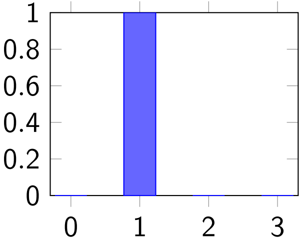
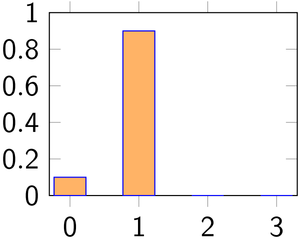
Deterministic P_{det} (n=1)

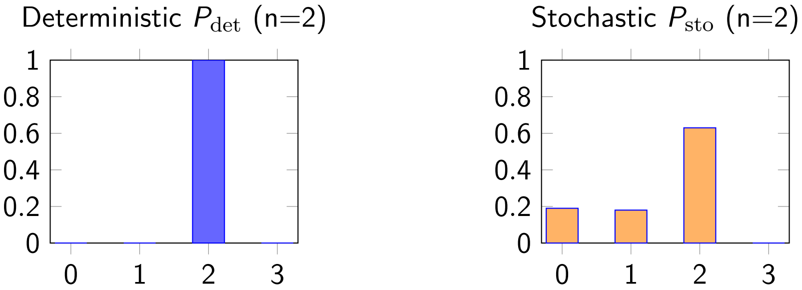


Stochastic P_{sto} (n=1)



Time evolution: $n = 2$

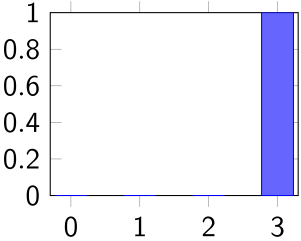
$$\boldsymbol{\pi}^{(2)} = \boldsymbol{\pi}^{(1)}\,\mathbf{P},$$



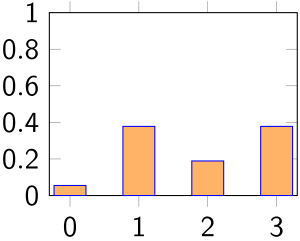
Time evolution: $n = 3$

$$\boldsymbol{\pi}^{(3)} = \boldsymbol{\pi}^{(2)}\,\mathbf{P},$$

Deterministic P_{det} (n=3)

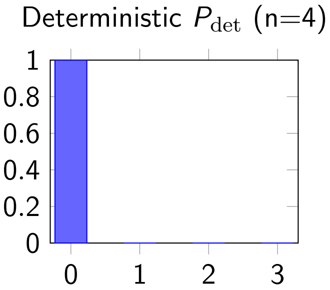


Stochastic P_{sto} (n=3)



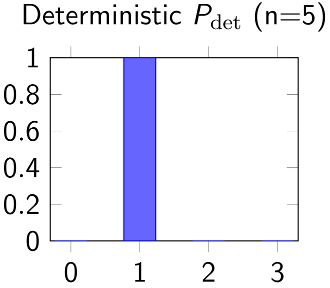
Time evolution: $n = 4$

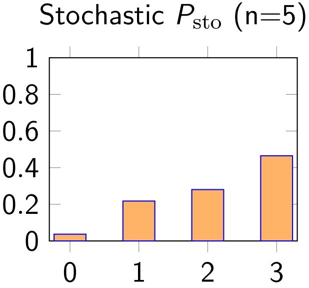
$$\boldsymbol{\pi}^{(4)} = \boldsymbol{\pi}^{(3)}\,\mathbf{P},$$



Time evolution: $n = 5$

$$\boldsymbol{\pi}^{(5)} = \boldsymbol{\pi}^{(4)}\,\mathbf{P},$$





Remarks and interpretation

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- Even when the stochastic transitions favor forward motion (like our matrix), randomness creates dispersion and memory loss of the exact initial position.
- To model a true discrete Newton's second law (position & velocity), one would enlarge the state space to include velocity and obtain a deterministic permutation on the extended state space; marginalizing out velocity then produces a stochastic-looking chain.

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Markov Chains remind us that knowing *everything* is rare,
but understanding *how uncertainty evolves* is power.

"Chance is only the measure of our ignorance."

— Henri Poincaré "Chance", Part I, Chapter 4 of Science and Method.

End of Lecture Series on Probability Theory

MA2631, Fall 2025

Ready to experiment?

Time to take your basics and run, and fly high.
Thanks for sticking around!