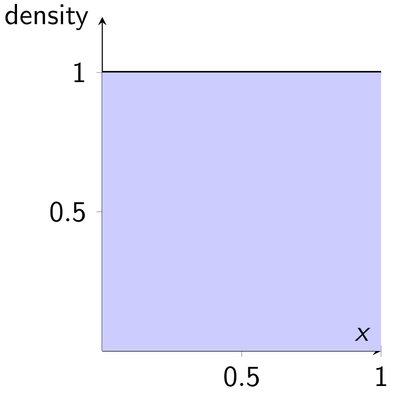
The Central Limit Theorem
Visualizing sums of Uniform(0,1)

MA 2631 Probability Theory Supplements

October 7, 2025

Sum of 1 Uniform(0,1) Variable

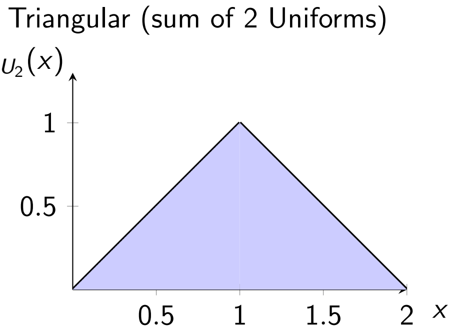
$$f_U(x) = 1, \quad x \in [0, 1].$$



Sum of 2 Uniform(0,1) Variables

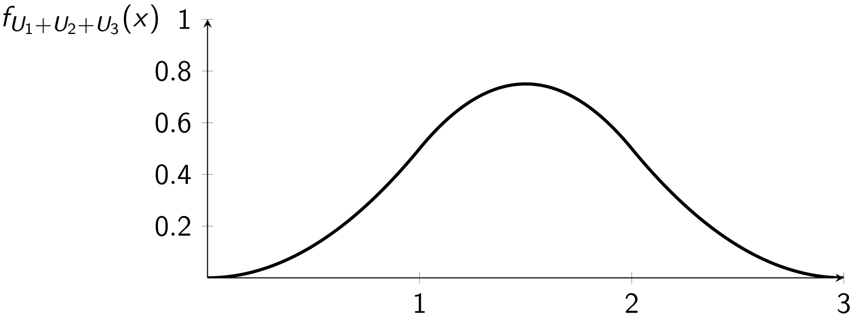
$$f_{U_1+U_2}(x) = \begin{cases} x, & x \in [0,1] \\ 2-x, & x \in [1,2] \\ 0, & \text{otherwise} \end{cases}$$

Why? The convolution theorem.



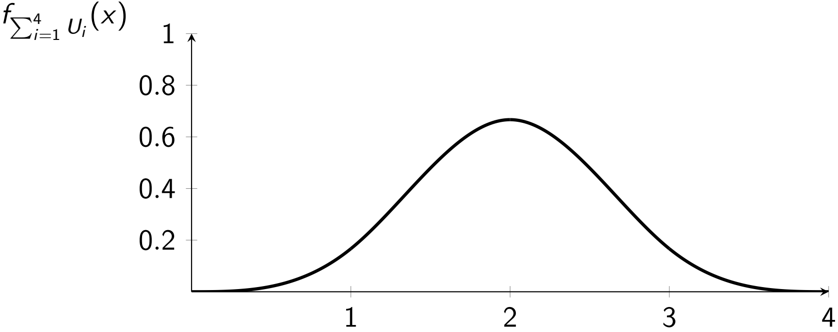
Sum of 3 Uniform(0,1) Variables

Still a piecewise density (believe it or not).

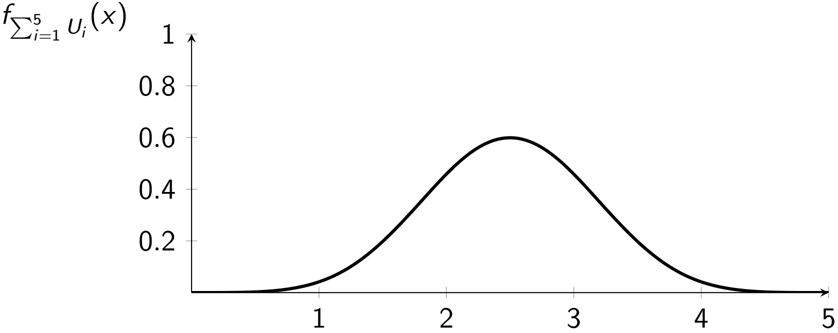


Sum of 4 Uniform(0,1) Variables

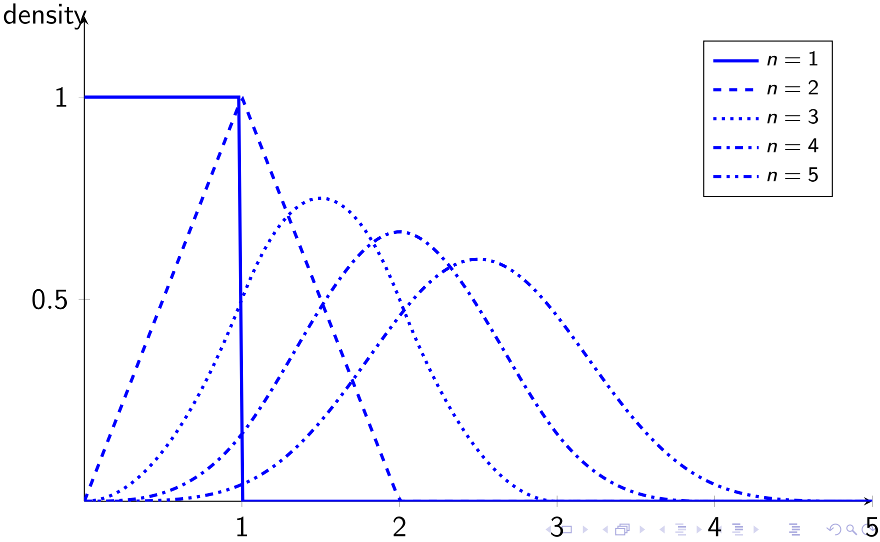
Note the increasing x range.



Sum of 5 Uniform(0,1) Variables



Sum of Uniform Random Variables $n = 1, \ldots, 5$



Centering and scaling (standardization)

Given $S_n = \sum_{i=1}^{n} U_i$ with $U_i \sim \mathrm{Uniform}(0,1)$ iid, the mean and variance are

$$\mathbb{E}[S_n] = \frac{n}{2}, \qquad \mathrm{Var}(S_n) = \frac{n}{12}.$$

Define the standardized sum

$$Z_n := \frac{S_n - \mathbb{E}[S_n]}{\sqrt{\mathrm{Var}(S_n)}} = \frac{S_n - n/2}{\sqrt{n/12}}.$$

The CLT says Z_n tends toward a standard normal distribution as $n \to \infty$.

Density via change of variable

Suppose the sum $S_n = \sum_{i=1}^{n} U_i$ has density $f_{S_n}(s)$. We defined the standardized sum

$$Z_n = \frac{S_n - n/2}{\sqrt{n/12}}.$$

Quick change-of-variable formula: If $Z_n = g(S_n)$, then the density of Z_n is

$$f_{Z_n}(z) = f_{S_n}(s)\left|\frac{ds}{dz}\right|, \qquad s = g^{-1}(z).$$

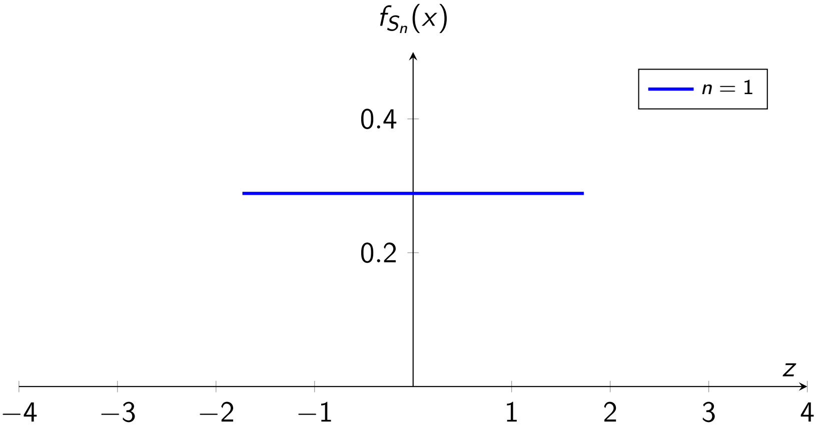
$$g(s) = \frac{s - n/2}{\sqrt{n/12}} \implies g^{-1}(z) = z\sqrt{n/12} + n/2, \quad \frac{ds}{dz} = \sqrt{n/12}.$$

$\Rightarrow$ Standardized density:

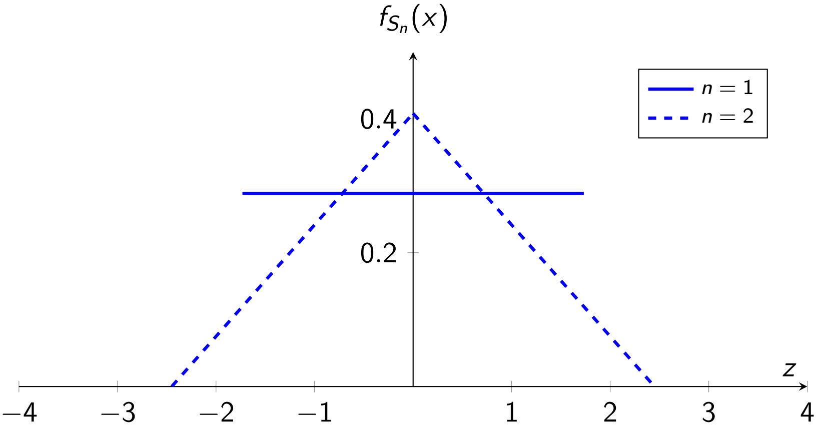
$$\boxed{f_{Z_n}(z) = f_{S_n}\left(z\sqrt{n/12} + n/2\right)\sqrt{n/12}}$$

This is the density version of the standardization step!

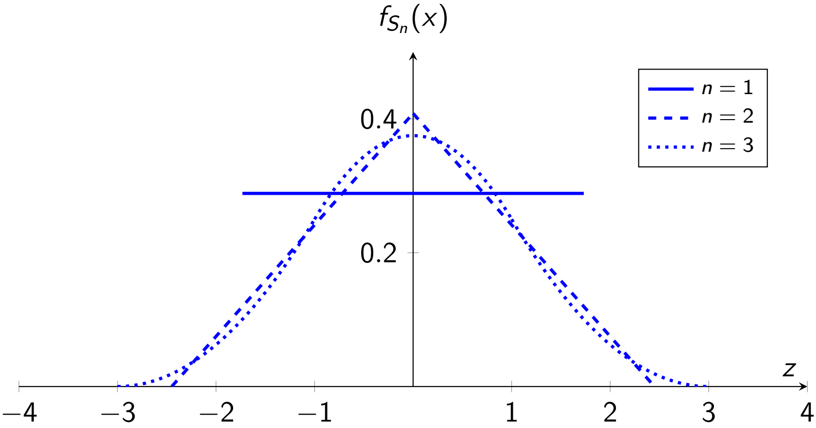
Standardized Irwin–Hall PDFs: $n = 1$



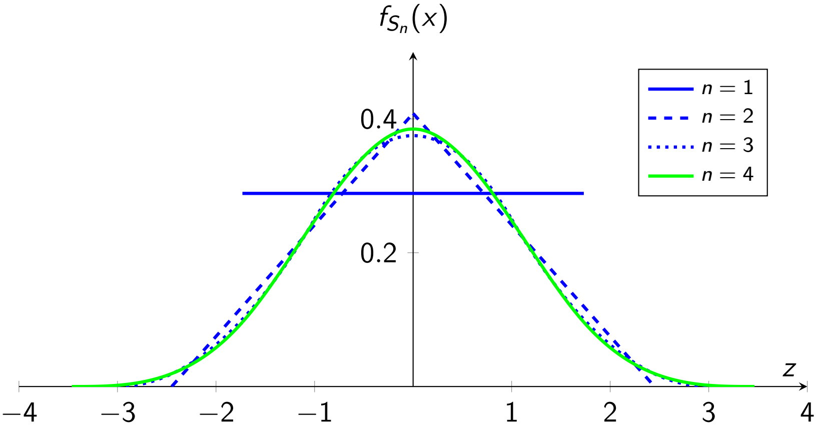
Standardized Irwin–Hall PDFs: $n = 2$



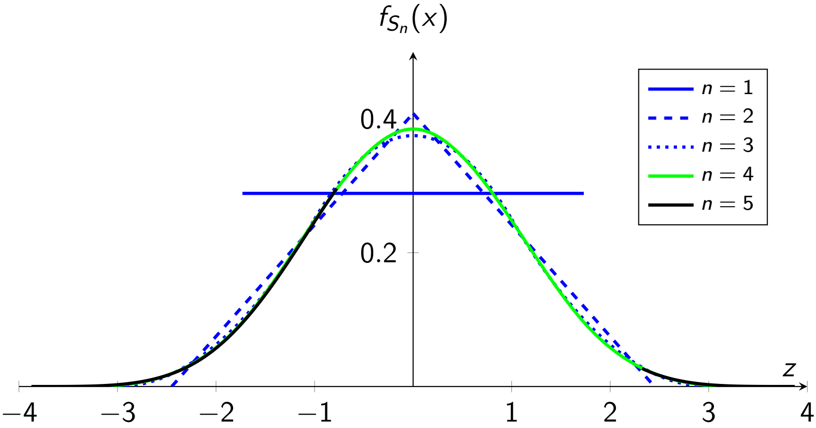
Standardized Irwin–Hall PDFs: $n = 3$



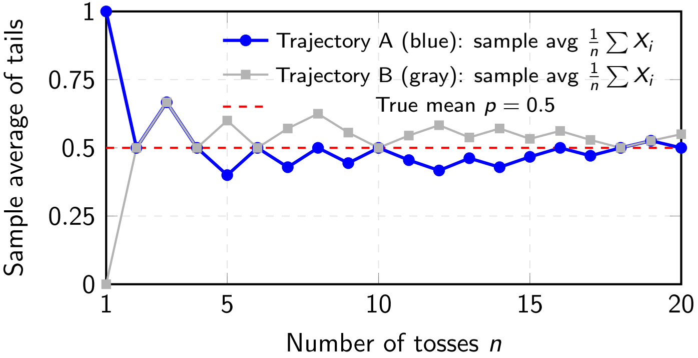
Standardized Irwin–Hall PDFs: $n = 4$



Standardized Irwin–Hall PDFs: $n = 5$



Weak Law of Large Numbers — two sample paths



Trajectory A (blue)
(1 = tail, 0 = head)
$\{1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0\}$

Trajectory B (gray)
(1 = tail, 0 = head)
$\{0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 1, 1\}$