# MA 1022 Calculus II Lecture Notes

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## Contents

# Part I
# Foundations of Integration

We introduce the definition of antiderivatives, indefinite integrals, Riemann sums and their limits – definite integrals, and lastly sweep through all integration techniques.

First, we provide a little motivation on why we want to study integration, a term we have not yet defined.

## Motivation

When you set your vehicle on cruise control at 50 miles per hour and travel for 4 hours, it is very easy to compute how far you have gone.

$$50\,\frac{\text{miles}}{\text{hour}} \times 4\,\text{hours} = 200\,\text{miles}.$$

If you happen to drive at two distinct velocities, say 50 mph for 1 hour and 20 mph for 3 hours, it is still reasonably easy to compute total displacement, i.e.,

$$50\,\frac{\text{miles}}{\text{hour}} \times 1\,\text{hour} + 20\,\frac{\text{miles}}{\text{hour}} \times 3\,\text{hour} = 110\,\text{miles}.$$

However, when you begin to drive in the city, your velocity is no longer a (piecewise) constant function of time. How do you find the total distance you have traveled?

You may record the instantaneous velocities $v(t_1), v(t_2), \ldots, v(t_n)$ over the time horizon $[t_1, t_n]$. For each subinterval $[t_i, t_{i+1}]$, you assume that the velocity is constant at $v(t_i)$, for $i = 1, \ldots, n-1$. Thus, total displacement, up to time $t_n$, satisfies

$$s(t_n) = v(t_1)(t_2 - t_1) + v(t_2)(t_3 - t_2) + \cdots + v(t_{n-1})(t_n - t_{n-1}).$$

To make life simpler, let's assume that the length of the subintervals is fixed, that is,

$$t_{i+1} - t_i = \Delta t,$$

which corresponds to the frequency at which you record velocity, i.e., every $\Delta t$ seconds.

Thus, total displacement up to time $t_n$ can be rewritten as

$$s(t_n) = [v(t_1) + v(t_2) + \cdots + v(t_{n-1})]\,\Delta t.$$

One would imagine that if we record the velocity **more frequently**, that is, by shrinking $\Delta t$ and incidentally increasing $n$, the number of recorded velocity data, we should be making better judgment of the total displacement.

So, a natural question arises. Does the limit

$$\lim_{\substack{\Delta t \to 0 \\ n \to \infty}} s\left(t_n\right) = \lim_{\substack{\Delta t \to 0 \\ n \to \infty}} \left[v\left(t_1\right) + v\left(t_2\right) + \cdots + v\left(t_{n-1}\right)\right] \Delta t$$

exist? If so, is it truly the total displacement? The limit is not entirely trivial because even though $\Delta t$ is shrinking, the number of terms you add is growing. You know the limit must be **finite** because you aren't driving towards the endless universe.

Set this technical idea aside. What does this have to do with Calculus? What is Calculus even to begin with? Are you reconsidering your life choices after taking Calculus I and still yet unable to answer this question?

Let's begin with actual CALCULUS! Historically, integration came first.

# 1 Lecture I: Antiderivatives and Indefinite Integrals

Given that you know the derivative $f'(x)$ of a function $f(x)$ in the natural domain of $f(x)$, can you recover the original function $f(x)$? If not, what do you need to recover it?

**Definition 1.1** (Antiderivative)**.** *A function $F$ is an **antiderivative** of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x \in I$.*

**Remark 1.1.** *$F$ is not unique for each $f$.*

**Theorem 1.1.** *If $F$ is an antiderivative of $f$ on an interval $I$, then the most general antiderivative of $f$ on $I$ is*

$$F(x) + C$$

*where $C$ is an arbitrary constant.*

**Remark 1.2.** *This is solely due to the fact that the derivative of a constant is zero.*

**Example 1.1.** Find an antiderivative, $F'(x)$, of the function $f(x) = 3x^2$ that satisfies $F(1) = -1$.

**Solution.** The general antiderivative is

$$F(x) = x^3 + C.$$

Using the condition $F(1) = -1$, we have

$$-1 = F(1) = (1)^3 + C \implies C = -2 \implies F(x) = x^3 - 2.$$

The following table provides the typical antiderivatives we use. The antidifferentiation rules we have invoked are merely the reverse process of the associated differentiation rules, mostly on Reverse Power Rule and Chain Rule.

| Function | General Antiderivative | Antidifferentiation Rule |
|---|---|---|
| $x^n$ | $\frac{1}{n+1}x^{n+1} + C$, $n \neq -1$ | Reverse Power Rule |
| $\sin(kx)$ | $-\frac{1}{k}\cos(kx) + C$ | Reverse Chain Rule |
| $\cos(kx)$ | $\frac{1}{k}\sin(kx) + C$ | Reverse Chain Rule |
| $\sec^2(kx)$ | $\frac{1}{k}\tan(kx) + C$ | Reverse Chain Rule |
| $-\csc^2(kx)$ | $\frac{1}{k}\cot(kx) + C$ | Reverse Chain Rule |
| $\sec(kx)\tan(kx)$ | $\frac{1}{k}\sec(kx) + C$ | Reverse Chain Rule |
| $-\csc(kx)\cot(kx)$ | $\frac{1}{k}\csc(kx) + C$ | Reverse Chain Rule |
| $e^{kx}$ | $\frac{1}{k}e^{kx} + C$ | Reverse Chain Rule |
| $\frac{1}{x}$ | $\ln\lvert kx\rvert + C$, $x \neq 0$ | Reverse Chain Rule |
| $\frac{1}{\sqrt{1-(kx)^2}}$ | $\frac{1}{k}\arcsin(kx) + C$ | Reverse Chain Rule |
| $\frac{1}{1+(kx)^2}$ | $\frac{1}{k}\arctan(kx) + C$ | Reverse Chain Rule |

**Remark 1.3** (Important). *You can **always** check if you have the correct antiderivative – just differentiate the antiderivative and see whether you get the original function back.*

**Proposition 1.1** (Antidifferentiation is linear). *If $F(x)$ and $G(x)$ are the antiderivatives of $f(x)$ and $g(x)$, respectively, then for constants $a$ and $b$, the antiderivative of $af(x) \pm bg(x)$ is*

$$aF(x) \pm bG(x).$$

Now, you may notice that we kept writing the English work "antiderivative", which is taking a lot of time and space. This pain gives rise to the following definition.

**Definition 1.2** (Indefinite Integrals). *The collection of all antiderivatives of $f$ is called the **indefinite integral** of $f$ with respect to $x$, and is denoted by*

$$\int f(x)\,dx.$$

*The symbol $\int$ is an **integral sign**. The function $f(x)$ is the **integrand** of the integral, and $x$ is the **variable of integration**.*

**Remark 1.4.** *Finding the antiderivative of $f$ and the indefinite integral of $f$ are precisely the same procedure and can be used interchangeably.*

**Example 1.2.** Find the general antiderivative of the following functions:

1. $f(x) = x^5$.

   **Solution.**
   $$F(x) = \frac{1}{5+1}x^{5+1} = \frac{1}{6}x^6 + C.$$

2. $g(x) = 1/\sqrt{x}$.

   **Solution.** First, we write $g(x) = x^{-1/2}$. Then,
   $$G(x) = \frac{1}{-\frac{1}{2}+1}x^{-\frac{1}{2}+1} = 2x^{\frac{1}{2}} + C.$$

3. $h(x) = \sin(2x)$.

   **Solution.**
   $$H(x) = -\frac{1}{2}\cos 2x + C.$$

4. $f(x) = \cos(x/2)$.

   **Solution.**
$$P(x) = 2\sin\left(\frac{x}{2}\right) + C.$$

5. $f(x) = e^{-3x}$.

   **Solution.**
$$Q(x) = -\frac{1}{3}e^{-3x} + C.$$

**Example 1.3.** Evaluate the following indefinite integrals:

1. $\int (x^2 - 2x + 5)\,dx$.

   **Solution.**
$$\int \left(x^2 - 2x + 5\right) dx = \frac{1}{3}x^3 - x^2 + 5x + C.$$

2. $\int \left(3e^{2x} + \frac{1}{2x} - \pi\right) dx$.

   **Solution.**
$$\int \left(3e^{2x} + \frac{1}{2x} - \pi\right) dx = \frac{3}{2}e^{2x} + \frac{1}{2}\ln|x| - \pi x + C.$$

Since finding the antiderivative and evaluating an indefinite integral are precisely the same thing, the indefinite integral inherits the linear property, i.e.,

$$\int (af(x) \pm bg(x))\,dx = \left(a\int f(x)\,dx\right) \pm \left(b\int g(x)\,dx\right) \tag{1}$$

where $a, b$ are constants.

**There is ABSOLUTELY NO such rule as**

$$\int f(x)\,g(x)\,dx = \left(\int f(x)\,dx\right)\left(\int g(x)\,dx\right)$$

# 2 Lecture II: Approximating Areas and Sigma Notations

There are various quests in classical geometry that involve formulas of areas and volumes of shapes. For example, the formula for the area of a circle came from Archimedes by inscribing polygons inside a circle. He computed the area of the polygons for each choice of number of sides and then found a pattern. He then took a limit as the number of sides goes to infinity. At the time, $\pi$ is not known. Archimedes' formula for the circle is $A = \frac{1}{2}Cr$ where $C$ is the circumference of the circle, and his proof relies on comparing the area of a circle (unknown at the time) to the area of a triangle with base $C$ and height $r$. In modern day understanding, this area formula certainly works since $C = 2\pi r$.

So, why integration, now? What really is integration? Archimedes' approach to the area of the circle, in fact, involves the area of a polygon. The area of a polygon involves adding up lots of identical triangles. Each of these triangles is an isosceles, with sides equal to the radius $r$. As we increase the number of sides, the triangle's height and base changes while the side length is preserved to be $r$.

We are used to adding things up **discretely** as the objects we add are **countable**. Integration is a concept of adding things up, **continuously**. In other words, you are adding very little things up, each with infinitesimal (varying according to some law/function) size. This makes computing an area a *special case* of integration. Whenever you need to add very little things up, you think of integration.

**Example 2.1** (Scientific processes involving integration)**.**

1. Area, volume, arc length, etc.
2. Displacement of a continuously moving object with time-varying velocity.
3. Kinetic energy of a continuously moving object with time-varying velocity and mass.
4. Mass of an object with spatially inhomogeneous density.
5. Probability of an event characterized by a range of values governed by some continuous density function.

## 2.1 Left and Right Endpoint Approximations

In the examples, we see that there is always something about varing quantities it is captured by a function $f(x)$. Here, we consider a concrete example.

**Example 2.2.** Find the area below the graph of the function $f(x) = \sqrt{4 - x^2}$ on $[0, 2]$ by approximating it with two vertical slabs.

**Solution.** We know the exact solution here, i.e., the area of a sector with 90 degrees and a radius of 2,

$$A = \frac{1}{4}\pi (2)^2 = \pi \approx 3.14159.$$

First, we approximate the area by **two** vertical slabs with an identical width of 1, with height decided by the function value on the **left** endpoint. We find that the sum of the area of the two bars is

$$L_2 = \Delta x \times [f(0) + f(1)] = 1 \times \left[\sqrt{4} + \sqrt{4 - 1^2}\right] = 2 + \sqrt{3} \approx 3.73205.$$

Hmmm. Gross overestimate. Can we improve by using **four** slabs with identical width of $\frac{1}{2}$? We find

$$\begin{aligned} L_4 &= \Delta x \times \left[f(0) + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right)\right] \\ &= 1 \times \left[\sqrt{4} + \sqrt{3.75} + \sqrt{3} + \sqrt{1.75}\right] \\ &\approx 3.49571. \end{aligned}$$

Getting better. Let's be finer with **eight** slabs with identical width $\frac{1}{4}$.

$$\begin{aligned} L_8 &= \Delta x \times \left[f(0) + f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right) + f(1) + f\left(\frac{5}{4}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{7}{4}\right)\right] \\ &= \frac{1}{4} \times (\text{STUFF}) \\ &\approx 3.33982 \end{aligned}$$

This is getting much closer than what we started with, relatively speaking.

No one says that we cannot do the **right endpoint approximation** – it is logically equivalent to the left endpoint approximation. Using **four** subintervals but now evaluate the function at the **right endpoints** of the subintervals, we find

$$R_4 = \Delta x \times \left[f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2)\right] \approx 2.49571.$$

We seem to have found a **lower bound** for the area estimate.

**Remark 2.1.** *However, in reality, for a general function, we do NOT necessarily know the true solution. We are merely refining the partition. In the example above, using the left endpoint approximation, we find the numbers* ***decreasing*** *towards the true answer. But if we don't know the true answer, we certainly don't want this decreasing sequence to reach negative infinity. Luckily, the right endpoint approximation provided such lower bound, ensuring that the true answer must lie between itself and the upper bound, provided by the left endpoint rule.*

**Remark 2.2.** *Note that the function itself is decreasing. This fact has caused the left endpoint approximation to be an overestimate, and the right endpoint approximation to be an underestimate.*

We will revisit this topic in more generality after we introduce the famous notation: the sigma notation to facilitate faster communication mathematically.

## 2.2 Sigma Notations Primer

Now, enough of this. We seem to be spending quite some time writing down the $L_4$ and $R_8$'s. It is cumbersome and inefficient. Let me reintroduce the lazy mathematicians.

**Definition 2.1** (Sigma Notation)**.** *Given a sequence of numbers $x_1, x_2, \ldots, x_n$, the sum of these numbers can be written as*

$$(x_1 + x_2 + \cdots + x_n) = \sum_{i=1}^{n} x_i$$

*where $x_i$ is the* ***summand****, $i$ is the* ***index symbol****, 1 is the* ***starting index****, and $n$ is the* ***ending index****; the symbol $\Sigma$ reads, "sigma", represents, sum.*

We will see the use of this in the context of Riemann sums, the precursor to the limit definition of a definite integral. Next class, we will first explore to calculate areas under the curve of known geometry, without the complication of sigma notations. Then, when tackling problems without geometric inspiration, we proceed with the sigma notation by noting a few simple sums first, such as sum of consecutive integers.

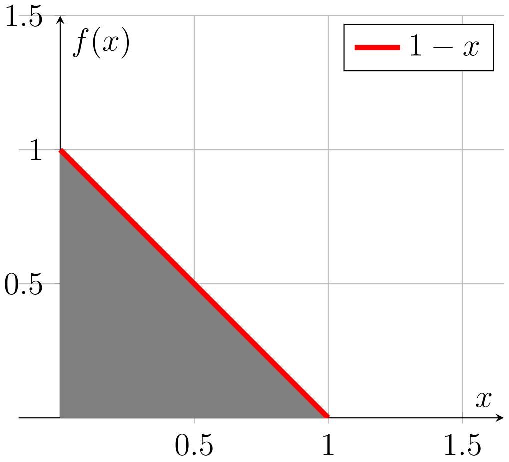
# 3 Lecture III: Definite Integrals and Riemann Sums

## 3.1 Definite Integral via Geometry

**Definition 3.1** (Definite Integrals: Geometric)**.** *Given a continuous function $f(x)$ on $[a,b]$, we say its definite integral from $a$ to $b$, is the area under the graph of $f$ over the interval, written as*

$$A = \int_a^b f(x)\, dx.$$

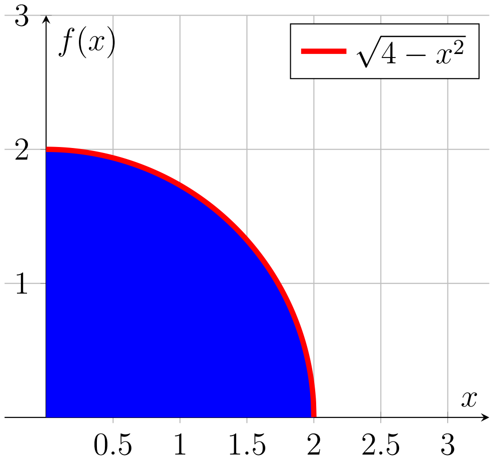
**Example 3.1.** Consider the function $y = f(x) = 1 - x$.




The shaded shape is a triangle with a height of 1 and a base of 1. The area is 1/2. More precisely,

$$\int_0^1 (1-x)\, dx = \frac{1}{2}.$$

**Example 3.2.** Consider the function $y = f(x) = \sqrt{4 - x^2}$ (same example as in the last lecture).

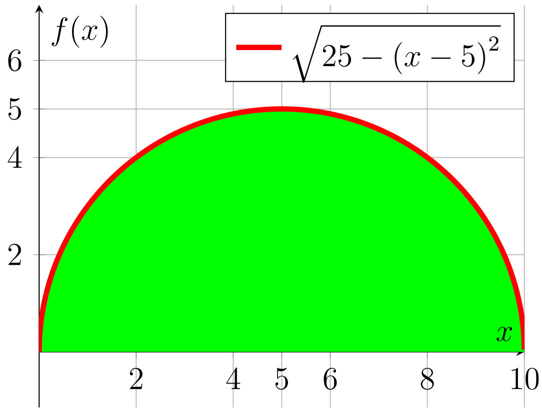



The shaded shape is a sector with a radius 2 and angle 90 degrees. The area is $\frac{1}{4}\pi(2)^2 = \pi$. More precisely,

$$\int_0^2 \sqrt{4-x^2}dx = \pi.$$

**Example 3.3.** Consider the function $y = f(x) = \sqrt{25-(x-5)^2}$ on $[0,10]$. Note that this is a semicircle centered at $(5,0)$. To see this, we square both sides and find

$$y^2 = 25-(x-5)^2 \implies (x-5)^2 + y^2 = 25.$$

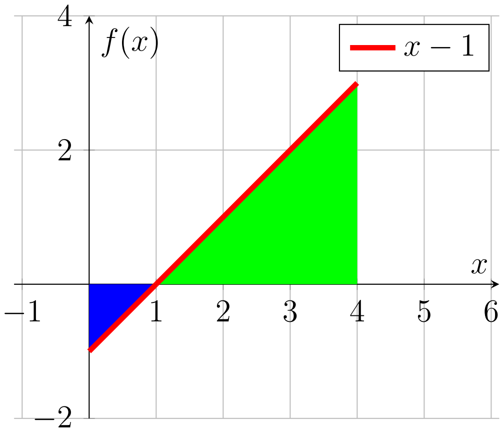



The shaded shape is a semicircle with a radius 5. The area is $\frac{1}{2}\pi(5)^2 = \frac{25}{2}\pi$. More precisely,

$$\int_0^{10} \sqrt{25-(x-5)^2}dx = \frac{25}{2}\pi.$$

We also allow negative area when the function is negative across the given domain.

**Example 3.4.** Consider the function $y = f(x) = x - 1$ on $[0, 4]$.




The shaded shape contains two triangles, one in blue underneath the $x$-axis, one in green above the $x$-axis.

The blue triangle has a **negative** area of $-\frac{1}{2}$, while the green triangle has an area of $\frac{9}{2}$.

$$\int_0^4 (x-1)\,dx = -\frac{1}{2} + \frac{9}{2} = 4.$$

You may ask why "negative area" shows up here or whether it makes sense at all. The short answer is that functions sometimes take negative values. But we haven't truthfully used function values in all examples above, but just the geometry of the shaded area. The true (and longer) answer, in fact, depends on the true definition of a definite integral, rather than the geometric one just provided. To arrive at the true definition, we set up some facts about sigma notations in preparation for adding infinitely many rectangles with very small widths.

## 3.2 Sigma Notations continued

**Definition 3.2** (Sigma Notation)**.** *Given a sequence of numbers $x_1, x_2, \ldots, x_n$, the sum of these numbers can be written as*

$$(x_1 + x_2 + \cdots + x_n) = \sum_{i=1}^{n} x_i$$

*where $x_i$ is the **summand**, $i$ is the **index symbol**, 1 is the **starting index**, and $n$ is the **ending index**; the symbol $\Sigma$ reads, "sigma", represents, sum.*

**Example 3.5** (The index symbol can change and it doesn't matter)**.**

1.

$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5$$

2.

$$\sum_{s=1}^{3} (-1)^s = (-1)^1 1 + (-1)^2 2 + (-1)^3 3 = -1 + 2 - 3$$

3.

$$\sum_{j=4}^{5} \frac{j^2}{j-1} = \frac{4^2}{4-1} + \frac{5^2}{5-1}$$

Carl Friedrich Gauss managed to add up the first hundred whole numbers, i.e.,

$$(1 + 2 + 3 + \cdots + 99 + 100) = \sum_{k=1}^{100} k$$

by observing that $k + (101 - k) = 101$ holds for every $k = 1, 2, \ldots, 50$. Thus, the sum is $101 \times 50 = 5050$. There are many versions of this story, but one thing stays true – he was seven when he figured this out.

**Proposition 3.1** (Properties of Sums)**.** *Given constants $a, b$, and two sequences $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$, we have*

1. *Sum and Difference*

$$\sum_{i=1}^{n} (x_i \pm y_i) = \left(\sum_{i=1}^{n} x_i\right) \pm \left(\sum_{i=1}^{n} y_i\right).$$

2. *Constant Multiple*

$$\sum_{i=1}^{n} a x_i = a \sum_{i=1}^{n} x_i$$

3. *Linear Combinations*

$$\sum_{i=1}^{n} (a x_i \pm b y_i) = a \left(\sum_{i=1}^{n} x_i\right) \pm b \left(\sum_{i=1}^{n} y_i\right)$$

4. *Adding up a constant*

$$\sum_{k=1}^{n} a = a \sum_{k=1}^{n} 1 = c \times (\textit{adding up the number } 1, \ n \textit{ times}) = an$$

**Example 3.6.** When you see a complicated-looking summand, yet its terms are all added or subtracted, you can divide and conquer,

$$\sum_{k=1}^{n} \left(3k - k^2\right) = \left(3\sum_{k=1}^{n} k\right) - \left(\sum_{k=1}^{n} k^2\right).$$

Hmmm, haven't I heard this before? Yes, you have. This is precisely the same property for Indefinite Integrals, see Equation (1).

You can also factor negative numbers,

$$\sum_{j=1}^{m} -\frac{1}{j^2} = -\sum_{j=1}^{m} \frac{1}{j^2}$$

## 3.3 Special Sums

We now consider three specific sums whose formula can be derived.

**Theorem 3.1** (Sum of Consecutive Positive Integers)**.**

$$\sum_{k=1}^{n} k = \frac{n\,(n+1)}{2}$$

*Proof.* Consider adding up twice and then dividing by 2, but in a particular order,

$$\begin{aligned} 1 + 2 + \cdots + n \\ n + (n-1) + \cdots + 1 \end{aligned}$$

Note that if we add up the terms, we obtain the same sum of $n+1$ for $n$ times. Thus, the sum of either sequence is as promised in the theorem. This argument is robust regardless of the parity of $n$. $\square$

We also have the sum of squares and cubes of consecutive integers,

$$\begin{aligned} \sum_{k=1}^{n} k^2 &= \frac{n\,(n+1)\,(2n+1)}{6} \\ \sum_{k=1}^{n} k^3 &= \left(\frac{n\,(n+1)}{2}\right)^2 = \left(\sum_{k=1}^{n} k\right)^2 \end{aligned}$$

where the formula for the sum of cubes is exactly the square of the sum of the first $n$ consecutive integers. This is a remarkable fact from arithmetic.

The proof requires *mathematical induction*. We unfortunately don't have the time to go into the details. Feel free to read up about it.

**Example 3.7.** Find the sum $\sum_{k=1}^{5} (k^2 - k)$.

**Solution.**

$$\begin{aligned}
\sum_{k=1}^{5} \left(k^2 - k\right) &\overset{\text{linearity}}{=} \left(\sum_{k=1}^{5} k^2\right) - \left(\sum_{k=1}^{5} k\right) \\
&\overset{\text{sum formula}}{=} \frac{5\,(5+1)\,(2\cdot 5+1)}{6} - \frac{5\,(5+1)}{2} \\
&= \frac{5\cdot 6\cdot 11}{6} - \frac{5\cdot 6}{2} \\
&= 55 - 15 \\
&= 40
\end{aligned}$$

This sum is nice because the starting index is 1, which is precisely what the sum formula gives. However, in practice, there may be cases where the starting index is NOT 1. What do we do then?

We may add the missing terms and then subtract it so that the starting index aligns with the formula.

**Example 3.8.**

$$\sum_{k=2}^{10} k = \left(\sum_{k=1}^{10} k\right) - 1 = \frac{10\,(10+1)}{2} - 1 = 55 - 1 = 54.$$

But what if the starting index is no longer 2, but say, some number quite large. Doing the add-subtract trick involves summing up the first few terms, which defeats the purpose. Instead, we employ the index shift trick, and then utilize expansions of the powers.

**Definition 3.3** (Index Shift)**.** *For a sequence $\{x_i\}$ where $i = 1, 2, \ldots, n$, the index shift property has*

$$\sum_{k=1}^{n} x_k = \sum_{k=0}^{n-1} x_{k+1} = \sum_{k=2}^{n+1} x_{k-1} = \ldots$$

The idea is to simply check whether the three sums represent precisely the same summand. After every shift, check if you maintain the same number of terms (here there are $n$ terms), and whether the first term and the last term align. For example, in the second sum, the first term is $x_{0+1} = x_1$, as expected, and the last term is $x_{n-1+1} = x_n$, again, as prescribed.

**Example 3.9.** Find the sum $\sum_{k=2}^{9} (k-1)^2$.

**Solution.** Now, you **must** note that the starting index is NOT 1, and therefore, you cannot apply the sum formula directly. In other words, you must align the starting index properly.

$$\sum_{k=2}^{9}(k-1)^2=\sum_{k=1}^{8}k^2=\frac{8\,(8+1)\,(2\cdot 8+1)}{6}=\frac{8\cdot 9\cdot 17}{6}=12\cdot 17=204.$$

**Example 3.10.** Find the sum $\sum_{k=5}^{15}k$.

**Solution.** You should NOT use the formula right away because the starting index is NOT 1. Instead, apply the index shift to make it happen,

$$\sum_{k=5}^{15}k=\sum_{k=1}^{11}(k+4)=\left(\sum_{k=1}^{11}k\right)+\left(4\sum_{k=1}^{11}1\right)=\frac{11\,(11+1)}{2}+4\times 11=66+44=110.$$

**Example 3.11.** Find the sum $\sum_{k=2}^{9}(k+1)^2$.

**Solution.** Now, again, you **must** note that the starting index is NOT 1, and therefore, you cannot apply the sum formula directly. In other words, you must align the starting index properly.

$$\begin{aligned}
\sum_{k=2}^{9}(k+1)^2 &\overset{\text{shift index}}{=} \sum_{k=1}^{8}(k+2)^2\\
&\overset{\text{expand}}{=} \sum_{k=1}^{8}\left(k^2+4k+4\right)\\
&\overset{\text{linearity}}{=} \left(\sum_{k=1}^{8}k^2\right)+\left(4\sum_{k=1}^{8}k\right)+\left(4\sum_{k=1}^{8}1\right)\\
&\overset{\text{sum formula}}{=} \frac{8\,(8+1)\,(2\cdot 8+1)}{6}+4\frac{8\,(8+1)}{2}+4\times 8\\
&= \frac{8\cdot 9\cdot 17}{6}+2\cdot 8\cdot 9+32\\
&= 204+144+32\\
&= 380
\end{aligned}$$

# 4 Lecture IV: The Definite Integral as the Limit of a Riemann Sum

## 4.1 Riemann Sums: Left and Right Endpoint Approximation Revisited

Equipped with the sigma notation, we can write down the left and right endpoint approximations more compactly.

**Definition 4.1** (Partition). *A set of points $P = x_i$ for $i = 0, 1, 2, \ldots, n$ with $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, which divides the closed interval $[a, b]$ into non-overlapping subintervals of the form, $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$, is called a* ***partition*** *of $[a, b]$. If the subintervals all have the same length, the set of points forms a* ***regular partition*** *of the interval $[a, b]$.*

We often refer to the set of $x_i$'s as **query points**.

**Definition 4.2** (Left and Right Endpoint Approximation).
*Left Endpoint Approximation:*

$$\begin{aligned} L_n &= f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x \\ &= [f(x_0) + \cdots + f(x_{n-1})]\Delta x \\ &= \Delta x \sum_{i=1}^{n} f(x_{i-1}) \end{aligned}$$

*Right Endpoint Approximation:*

$$\begin{aligned} R_n &= f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \\ &= [f(x_1) + \cdots + f(x_n)]\Delta x \\ &= \Delta x \sum_{i=1}^{n} f(x_i) \end{aligned}$$

**Remark 4.1.** *Notice that the* ***only*** *difference is index at which you evaluate. The "leftness" is captured by the index $i - 1$, while the "rightness" is captured by $i$.*

**Definition 4.3** (Riemann Sums). *Let $f(x)$ be defined on a closed interval $[a, b]$ and let $P$ be a regular partition of $[a, b]$. Let $\Delta x$ be the width of each subinterval $[x_{i-1}, x_i]$ and for each $i$, let $x_i^*$ be any point in $[x_{i-1}, x_i]$. A* ***Riemann Sum*** *is defined for $f(x)$ as*

$$\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

This definition makes the left and right endpoint approximations **special cases** of a Riemann sum. More precisely, on $[x_{i-1}, x_i]$, we have

$$\begin{array}{rl} \text{Left Endpoint Approximation} & x_i^* = x_{i-1} = a + (i-1)\,\Delta x \\ \text{Right Endpoint Approximation} & x_i^* = x_i = a + i\Delta x \end{array}$$

This precisely shows the "left" and "right" endpoints, which will help you remember exactly the rule if all you can recall is the **general form**, i.e., the Riemann sum.

## 4.2 Definite Integral as a Limit of Riemann Sums

**Definition 4.4** (Area under the curve)**.** *The area under the curve of a continuous, nonnegative function $y = f(x)$ on $[a, b]$ is the limit, if exists,*

$$A = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x,$$

*where the step size $\Delta x = \frac{b-a}{n}$. We also refer to*

$$Area\,(n) = \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

Depending on whether the function is increasing or decreasing, the Riemann sum may produce an underestimate, called **lower sum**, or an overestimate, called **upper sum**. Having both the upper and lower sums ensures that the true answer lies between them.

One particular difference between the approximations we made in the last lecture and the limit of Riemann sums is that the former uses a finite number of subintervals, while the latter uses infinitely many. In fact, the limiting procedure is the mathematical glue that links the two symbols $\Sigma$ and $\int$.

In the following examples, we aim to develop a general procedure of forming Riemann sums for specific functions over specific intervals. Once we gain enough proficiency, we will begin to consider the limiting case.

**Example 4.1.** Find a lower sum for $f(x) = 10 - x^2$ on $[1, 2]$ with 4 subintervals.

**Solution 4.1.** On this interval, $f(x)$ is decreasing, which means a lower sum can be achieved by the Right Endpoint Approximation (as a Riemann sum).

The first step is to list all the **query points**,

$$\begin{aligned}
x_0 &= 1 \\
x_1 &= 1.25 \\
x_2 &= 1.5 \\
x_3 &= 1.75 \\
x_4 &= 2
\end{aligned}$$

The second step is to identify the step size $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$.
We now assemble the Riemann sum using the Right Endpoint Approximation,

$$\begin{aligned}
R_4 = \sum_{i=1}^{4} f(x_i)\,\Delta x &= \frac{1}{4}\left[(10 - f(1.25)) + (10 - f(1.5)) + (10 - f(1.75)) + (10 - f(2))\right] \\
&= \frac{1}{4}\left[\left(10 - 1.25^2\right) + \left(10 - 1.5^2\right) + \left(10 - 1.75^2\right) + \left(10 - 2^2\right)\right] \\
&= 7.28
\end{aligned}$$

One may quickly formulate an **upper sum** of the area of interest above, using $L_4$,

$$\begin{aligned}
L_4 = \sum_{i=1}^{4} f(x_{i-1})\,\Delta x &= \frac{1}{4}\left[(10 - f(1)) + (10 - f(1.25)) + (10 - f(1.5)) + (10 - f(1.75))\right] \\
&= \frac{1}{4}\left[\left(10 - 1^2\right) + \left(10 - 1.25^2\right) + \left(10 - 1.5^2\right) + \left(10 - 1.75^2\right)\right] \\
&= 8.03125
\end{aligned}$$

Now, we can focus on the limiting case, which requires some general thought process.

**Example 4.2** (Limit of a Riemann Sum using Right-endpoint Approximation)**.** Continue with the same function on the same interval from the last example. Consider a **partition** of the interval $[1, 2]$ with $n$ subintervals. Each subinterval has length

$$\Delta x = \frac{b-a}{n} = \frac{1}{n}.$$

The set of query points would be

$$P = \left\{1, 1 + \frac{1}{n}, 1 + \frac{2}{n}, \ldots, 1 + \frac{n-1}{n}, 2\right\},$$

where each element corresponds to $x_0, x_1, x_2, \ldots, x_n$, respectively.

In order to assemble the Riemann sum with $n$ subintervals, it is useful to write an arbitrary query point in terms of $n$ and $i$. Here, we realize that

$$x_i^* = 1 + i\Delta x = 1 + \frac{i}{n}$$

with 1 being the left-most endpoint. Each query point follows 1 by multiples of subinterval size $\Delta x$.

For the **right-endpoint approximation**, we use all query points but the first, so

$$x_i^* = 1 + i\Delta x = 1 + \frac{i}{n}, \quad i = 1, 2, 3, \ldots, n.$$

In the following, we demonstrate the right-endpoint approximation. The approximate area, as a function of $n$, satisfies,

$$\begin{aligned}
Area\,(n) &= \sum_{i=1}^{n} f\,(x_i)\,\Delta x, \\
&\overset{\text{factor out }\Delta x}{=} \Delta x \sum_{i=1}^{n} f\,(1 + i\Delta x) \\
&\overset{f(x)=10-x^2}{=} \Delta x \sum_{i=1}^{n} \left(10 - (1 + i\Delta x)^2\right) \\
&\overset{\text{Expand}}{=} \Delta x \sum_{i=1}^{n} \left(10 - \left(1 + 2i\Delta x + (\Delta x)^2\, i^2\right)\right) \\
&\overset{\text{Simplify}}{=} \Delta x \sum_{i=1}^{n} \left(9 - 2i\Delta x - (\Delta x)^2\, i^2\right) \\
&\overset{\text{Linearity of sums}}{=} \Delta x \left[\left(\sum_{i=1}^{n} 9\right) - \left(2\Delta x \sum_{i=1}^{n} i\right) - (\Delta x)^2 \sum_{i=1}^{n} i^2\right] \\
&\overset{\text{Sum Formulas}}{=} \Delta x \left[9n - 2\Delta x \frac{n\,(n+1)}{2} - (\Delta x)^2\, \frac{n\,(n+1)\,(2n+1)}{6}\right] \\
&= \frac{1}{n}\left[9n - \frac{2}{n}\frac{n\,(n+1)}{2} - \frac{1}{n^2}\frac{n\,(n+1)\,(2n+1)}{6}\right] \\
&\overset{\text{Simplify}}{=} 9 - \frac{n+1}{n} - \frac{n\,(n+1)\,(2n+1)}{6n^3}
\end{aligned}$$

By definition of the area under the curve, we have

$$\begin{aligned}
A = \int_1^2 \left(10 - x^2\right) dx &= \lim_{n\to\infty} Area\,(n) \\
&= \lim_{n\to\infty} \left[9 - \frac{n+1}{n} - \frac{n\,(n+1)\,(2n+1)}{6n^3}\right] \\
&= 9 - 1 - \frac{1}{3} \\
&= \frac{23}{3} \approx 7.6667
\end{aligned}$$

which indeed lies between the upper and lower sum.

**Example 4.3** (Limit of a Riemann Sum using Left-endpoint Approximation)**.** This example is avoided in class due the pedagogical inconvenience induced by the ugly calculations. However, the principles are the same, and you should at least read through to learn of how the details are resolved step-by-step.

We use the same example as above, but now use the **left-endpoint approximation**. The **only** difference lies in the query points we select. In this case, we use every point *but* the last point. Mathematically, for the left-endpoint approximation, we use all query points but the last, so

$$x_i^* = 1 + (i-1)\,\Delta x = 1 + \frac{i-1}{n}, \quad i = 1, 2, 3, \ldots, n.$$

Thus, the Riemann sum with left-endpoint approximation is given by

$$
\begin{aligned}
Area\,(n) &= \sum_{i=1}^{n} f\left(x_{i-1}\right)\Delta x, \\
&\overset{\text{Left-endpoint}}{=} \Delta x \sum_{i=1}^{n} f\left(1+\frac{i-1}{n}\right) \\
&\overset{f(x)=10-x^2}{=} \Delta x \sum_{i=1}^{n}\left(10-\left(1+\frac{i-1}{n}\right)^2\right) \\
&\overset{\text{Expand}}{=} \Delta x \sum_{i=1}^{n}\left(10-\left(1+\frac{2\,(i-1)}{n}+\frac{(i-1)^2}{n^2}\right)\right) \\
&\overset{\text{Linearity of sums}}{=} \Delta x\left[\left(\sum_{i=1}^{n} 9\right)-\left(\frac{2}{n}\sum_{i=1}^{n}(i-1)\right)-\frac{1}{n^2}\sum_{i=1}^{n}(i-1)^2\right] \\
&\overset{\text{Linearity of sums}}{=} \Delta x\left[9n-\frac{2}{n}\left(\sum_{i=1}^{n} i-\sum_{i=1}^{n} 1\right)-\frac{1}{n^2}\left(\sum_{i=1}^{n} i^2-2\sum_{i=1}^{n} i+\sum_{i=1}^{n} 1\right)\right] \\
&\overset{\text{Sum Formulas}}{=} \Delta x\left[9n-\frac{2}{n}\left[\frac{n\,(n+1)}{2}-n\right]-\frac{1}{n^2}\left[\frac{n\,(n+1)\,(2n+1)}{6}-2\frac{n\,(n+1)}{2}+n\right]\right] \\
&\overset{\text{Simplify}}{=} \frac{1}{n}\left[9n-(n+1-2)-\left(\frac{(n+1)\,(2n+1)}{6n}-\frac{n+1}{n}+\frac{1}{n}\right)\right] \\
&= 9-\frac{n-1}{n}-\frac{(n+1)\,(2n+1)}{6n^2}+\frac{1}{n}
\end{aligned}
$$

Taking a limit yields

$$
\begin{aligned}
\lim_{n\to\infty} Area\,(n) &= \lim_{n\to\infty} 9-\frac{n-1}{n}-\frac{(n+1)\,(2n+1)}{6n^2}+\frac{1}{n} \\
&= 9-1-\frac{1}{3}+0 \\
&= \frac{23}{3},
\end{aligned}
$$

the same as the right-endpoint approximation, as it should be.

## 4.3 Properties of Definite Integral

Since the definite integral is the limit of a Riemann sum, it satisfies all the properties of a sum, i.e., sum, difference, constant multiple, etc (see Prop. 3.1). There is one particular property one may further borrow from sums.

**Proposition 4.1** (Patching Property for Sums)**.**

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k,$$

*namely, you can split up the sum at whichever index $m$ you want.*

**Example 4.4.**

$$\begin{aligned}\sum_{k=1}^{5} k &= \sum_{k=1}^{2} k + \sum_{k=3}^{5} k \\ &= (1+2) + (3+4+5)\end{aligned}$$

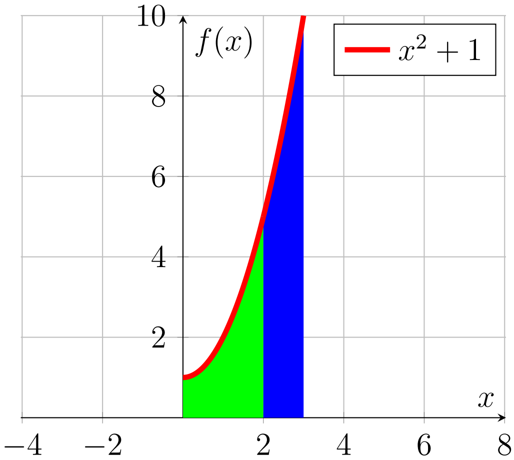
**Proposition 4.2** (Patching Property for Definite Integrals)**.** *For a continuous function $f(x)$ on $[a, b]$, we have, for every $c \in Dom(f)$,*

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

**Example 4.5.** Consider the function $f(x) = x^2 + 1$ on $[0, 3]$.

$$\int_0^3 \left(x^2+1\right) dx = \int_0^2 \left(x^2+1\right) dx + \int_2^3 \left(x^2+1\right) dx$$

exemplified by green and blue area added together.




**Proposition 4.3** (Complementary Property)**.**

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx \tag{2}$$

*namely, the integral from $a$ to $b$ is the negative of the integral fomr $b$ to $a$.*

**Corollary 4.1.** *As a consequence of the Complementary Property (Prop. 4.3), we must have*

$$\int_a^a f(x)\,dx = 0$$

*Proof.* Let $b = a$ in Equation (2). We have

$$\int_a^a f(x)\,dx = -\int_a^a f(x)\,dx \implies 2\int_a^a f(x)\,dx = 0 \implies \int_a^a f(x)\,dx = 0$$

namely, the only number equal to its negative value is the number 0. □

# Part II

# Integration Techniques

## 5 Lecture V: Fundamental Theorem of Calculus

### 5.1 A Motivating Example

Last week, we concluded with an expression for the definite integral of a function $f(x)$ on $[a, b]$, via the limit of a Riemann sum,

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x, \quad \Delta x = \frac{b-a}{n}.$$

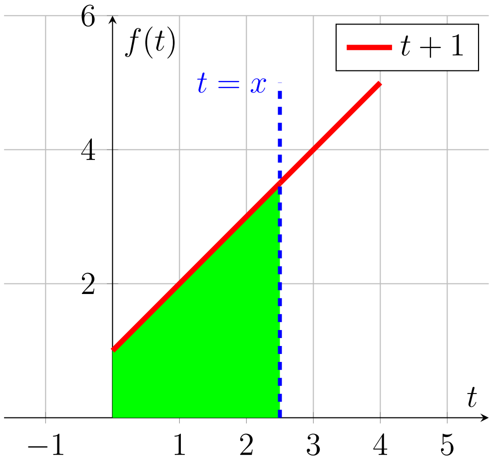
To set up the Riemann sum, we've used the left- or right-endpoint approximation. To evaluate the resulting sum, we've used the sum formula for consecutive integers, or sum of squares of consecutive integers. To evaluate the limit, we used ideas from Calculus I.

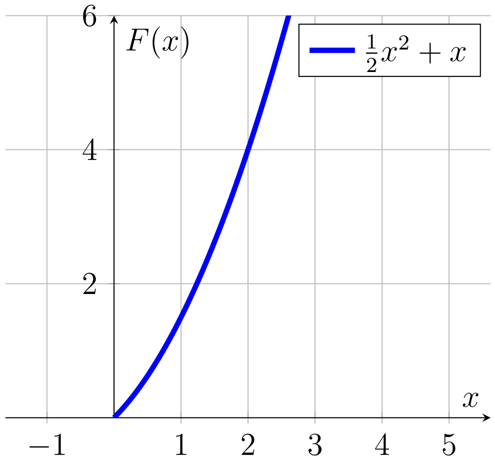
This procedure, while mathematically sound, is painful. In today's lecture, we aim to develop a systematic way of computing the definite integral without setting up a Riemann sum. We provide a powerful tool that links between definite integrals and antiderivatives.

The first idea is to define a function $F(x)$ that describes the area under $f(x)$ from a fixed point $a$ to an arbitrary point $x$,

$$F(x) = \int_a^x f(t)\, dt.$$

This means the definite integral of $f(x)$ on $[a, b]$ is just the function value $F(b)$.



**Example 5.1.** Consider the function $f(t) = t + 1$ on $[0, 4]$. Thus, for an arbitrary upper bound $x$,

$$F(x) = \int_0^x (t+1)\, dt \overset{\text{trapezoid}}{=} \frac{1 + (x+1)}{2} x = \frac{(x+2)\, x}{2} = \frac{1}{2} x^2 + x$$

because at $t = x$, the area under the curve of $f(t)$ is a trapezoid, with side lengths 1 and $x + 1$ and a width of $x$.

Now, if we look more closely at the integrand $f(x)$ and the defined function $F(x)$, we see that $F$ is an antiderivative of $f$, i.e.,

$$\begin{aligned} f(x) &= x + 1 \\ F(x) &= \frac{1}{2} x^2 + x \implies F'(x) = f(x). \end{aligned}$$

Certainly, $F(0) = 0$, as expected because at $t = 0$, there is no area to add!

**Is this a coincidence? Does this relationship hold for more general nonlinear $f(x)$?** Here comes the magical moment where differential and integral calculus join hands.

## 5.2 Fundamental Theorem of Calculus Part I

We first present a result that is precisely restating the Mean Value Theorem (from Calculus I), namely, the average value of function must be attained somewhere in an interval.

**Theorem 5.1** (Mean Value Theorem for Integrals)**.** *If $f(x)$ is continuous over an interval $[a, b]$, then there is at least one point $c \in [a, b]$ such that*

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

The way to understand this is to recall a little about the Mean Value Theorem from Calculus I. There, for a differentiable function $f(x)$ on $[a, b]$, the slope of the secant line must be achieved at least one time, valued $f'(c)$ for some $c \in (a, b)$. The slope of the secant line is also known as the **average rate of change**, or average velocity. Here, in the MVT for Integrals, it is just another way of stating the same *physical* concept that the average velocity (right-hand side) is attained somewhere at $x = c$ with value $f(c)$.

Equipped with the Mean Value Theorem for integrals, along with the observations made earlier in Example 5.1, we now present one of the most important ideas in Calculus.

**Theorem 5.2** (Fundamental Theorem of Calculus I (FTC I)). *If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by*

$$F(x) = \int_a^x f(t)\, dt,$$

*then $F'(x) = f(x)$ over $[a, b]$.*

*Proof.* We proceed directly by taking a derivative of $F(x)$, using the limit quotient definition.

$$\begin{aligned}
F'(x) &\overset{\text{definition of the derivative}}{=} \lim_{h\to 0} \frac{1}{h}\left[F(x+h) - F(x)\right] \\
&\overset{\text{definition of } F(x)}{=} \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt\right] \\
&\overset{\text{patching property, Proposition 4.1}}{=} \lim_{h\to 0} \frac{1}{h}\left[\int_x^{x+h} f(t)\, dt\right] \\
&\overset{\text{Mean Value Theorem for Integrals}}{=} \lim_{h\to 0} f(c) \\
&\overset{\text{Shrinking the interval } [x, x+h] \text{ bring all points to } x}{=} f(x)
\end{aligned}$$

□

**Remark 5.1.** *The essence is that the derivative of $F(x)$ is precisely the integrand with variable $x$. The lower bound $a$ does NOT matter.*

**Example 5.2** (Finding the Derivative with FTC I). Find the derivative of

$$g(x) = \int_1^x \frac{1}{t^3 + 1} dt$$

**Solution.**

$$f'(x) = \frac{1}{x^3 + 1}.$$

**Example 5.3** (Finding the Derivative with FTC I AND the Chain Rule). Find the derivative of

$$F(x) = \int_1^{\sqrt{x}} \sin(t)\, dt$$

**Solution.** We treat $F(x)$ as

$$F(u(x)) = \int_1^{u(x)} \sin(t)\, dt$$

where $u(x) = \sqrt{x}$. Define $f(x) = \sin(x)$. Thus, by FTC 1, we have

$$F'(x) = \frac{dF}{dx} \overset{\text{Chain Rule}}{=} \boxed{\frac{dF}{du}}\frac{du}{dx} \overset{\text{FTC I}}{=} \boxed{f(u)}\frac{du}{dx} = \sin(u)\frac{du}{dx} = \sin\left(\sqrt{x}\right)\frac{1}{2}x^{-\frac{1}{2}}.$$

**Example 5.4** (Finding the Derivative with Two Variable Limits of Integration)**.** Find the derivative of

$$F(x) = \int_x^{x^2} t^3 dt$$

**Solution.** We utilize the patching property, and then the Chain Rule (used in the last example).

$$\begin{aligned}
F(x) &= \int_x^{x^2} t^3 dt \\
&\overset{\text{Prop. 4.2}}{=} \int_x^c t^3 dt + \int_c^{x^2} t^3 dt \\
&\overset{\text{Prop. 4.3}}{=} -\int_c^x t^3 dt + \int_c^{x^2} t^3 dt \\
\implies F'(x) &= -x^3 + \left(x^2\right)^3 (2x) \\
&= 2x^7 - x^3
\end{aligned}$$

## 5.3 Fundamental Theorem of Calculus Part II

Now, you may say, cool, we just defined a new function $F(x) = \int_a^x f(t)\,dt$ and found its derivative as $f(x)$. So what? Is it going to help me evaluate definite integrals with a less messy method than doing Riemann sums?

Yes.

Consider the definite integral $\int_a^b f(x)\,dx$ where $F(x)$, defined as above, is the antiderivative of $f(x)$. Knowing that antiderivative is NOT unique, but a family of functions, we can say that $G(x) = F(x) + C$ is also an antiderivative of $f(x)$. Incidentally,

$$\begin{aligned}
G(b) - G(a) &= F(b) + C - (F(a) + C) \\
&= F(b) - F(a) \\
&= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\
&= \int_a^b f(t)\,dt
\end{aligned}$$

Note that this antiderivative $G$ is completely arbitrary, meaning that to evaluate the definite integral, it suffices to find **one** antiderivative and take a difference of the function values at the endpoints. This result is called the **Fundamental Theorem of Calculus Part II**, or also known as the **Evaluation Theorem**.

**Theorem 5.3** (Fundamental Theorem of Calculus Part II (FTC II))**.** *If $f$ is continuous over the interval $[a, b]$ and $F(x)$ is* ***any*** *antiderivative of $f(x)$, then*

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

*Proof.* See OpenStax Textbook, Volume II, Theorem 1.5. □

**Remark 5.2.** *It is absolutely crutical that you understand how integrals, derivatives and antiderivatives are related, and it is NOT obvious. The essence of their relationship is based on limits first, then some properties of the definite integral, and lastly, some properties of antiderivatives. Make a full story and tell it to yourself in the mirror.*

**Example 5.5.**

$$\begin{aligned}
\int_{-2}^{2} \left(t^2 - 4\right) dt &= \left[\frac{t^3}{3} - 4t\right]_{t=-2}^{t=2} \\
&= \left[\frac{2^3}{3} - 4 \cdot 2\right] - \left[\frac{(-2)^3}{3} - 4 \cdot (-2)\right] \\
&= \left(\frac{8}{3} - 8\right) - \left(-\frac{8}{3} + 8\right) \\
&= \frac{8}{3} - 8 + \frac{8}{3} - 8 \\
&= \frac{16}{3} - 16 \\
&= -\frac{32}{3}
\end{aligned}$$

**Example 5.6.**

$$
\begin{aligned}
\int_1^4 \frac{2-\sqrt{t}}{t^2} dt &= \int_1^4 \left[2t^{-2} - \frac{t^{1/2}}{t^2}\right] dt \\
&= \int_1^4 \left[2t^{-2} - t^{-\frac{3}{2}}\right] dt \\
&= \left[2\frac{t^{-1}}{-1} - \left(-2t^{-\frac{1}{2}}\right)\right]_{t=1}^{t=4} \\
&= \left[2t^{-\frac{1}{2}} - 2t^{-1}\right]_{t=1}^{t=4} \\
&= \left[2\left(4\right)^{-\frac{1}{2}} - 2\left(4\right)^{-1}\right] - \left[2\left(1\right)^{-\frac{1}{2}} - 2\left(1\right)^{-1}\right] \\
&= \left[2 \cdot \frac{1}{2} - \frac{1}{2}\right] - [2-2] \\
&= 1 - \frac{1}{2} - 0 \\
&= \frac{1}{2}
\end{aligned}
$$

# 6 Lecture VI: Integration Formulas and Application of the FTC

## 6.1 The Net Change Theorem

Let's recall the evaluation theorem, or the FTC part II,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

We know that $F'(x) = f(x)$, so we can make this formula self-contained by using expressions involving $F(x)$ alone. The result is the following theorem.

**Theorem 6.1** (Net Change Theorem)**.**

$$F(b) = F(a) + \int_a^b F'(x)\,dx$$

*or*

$$F(b) - F(a) = \int_a^b F'(x)\,dx$$

The significance of this theorem lies in the *physical* interpretation. If $F(x)$ is displacement, then one may consider $F(a)$ as the initial position at $x = a$, $F'(x)$ is velocity, which we integrate over to obtain accumulated displacement. Adding the initial position and the accumulated displacement, we obtain the final displacement $F(b)$.

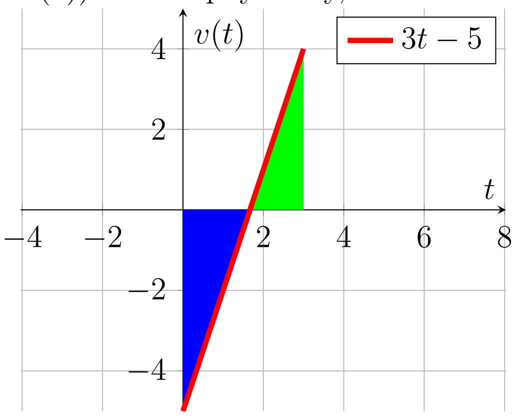
**Example 6.1** (Finding Net Displacement)**.** Given a velocity function $v(t) = 3t - 5$ (in meters per second) for a particle in motion from time $t = 0$ to time $t = 3$, find the net displacement of the particle.

**Solution.** Let $s(t)$ be the displacement at time $t$. We also know the relationship $s'(t) = v(t)$. Thus, applying the net change theorem, we integrate the velocity function from 0 to 3, i.e.,

$$\begin{aligned} s(3) - s(0) &= \int_0^3 v(t)\,dt \\ &= \int_0^3 (3t-5)\,dt \\ &= \left[\frac{3t^2}{2} - 5t\right]_{t=0}^{t=3} \\ &= \left[\frac{3}{2}(3)^3 - 5(3)\right] - [0+0] \\ &= -\frac{3}{2} \end{aligned}$$

The net displacement is $-\frac{3}{2}$ meters.

In the sketch, one see that the initial velocity is $v(0) = -5$ which means the particle is gaining negative displacement. It starts to come back as time goes on, attaining positive velocity for $t > 5/3$. The net displacement of $-\frac{3}{2}$ simply indicates that the particle has "gone" so negative that it is *just* making it back to its initial position (which would correspond to $s(0)$, or equivalently a net change of 0 meters from $s(0)$). Think physically, and then translate it mathematically.



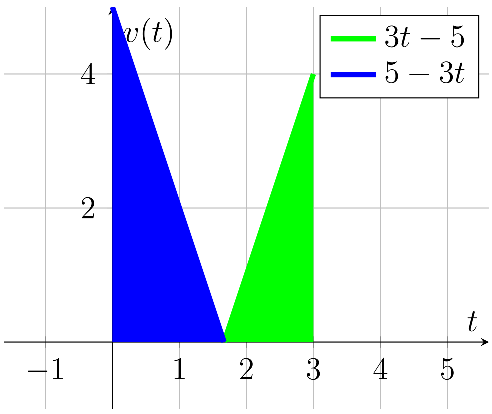

**Example 6.2** (Finding total distance traveled)**.** Displacement and distance are two different physical quantities. Distance is the accumulation of **speed**, which is the absolute value of velocity. In other words, "traveling at a positive **or** negative velocity" are both accruing positive distance. So, to find total distance traveled, we integrate $|v(t)|$ from $t = 0$ to $t = 3$.

The first step is to express $|v(t)|$,

$$|v(t)| = \begin{cases} -v(t), & 0 \le t \le \frac{5}{3}, \\ v(t), & \frac{5}{3} \le t \le 3. \end{cases}$$

Then, the total distance traveled is given by

$$\begin{aligned}
\int_0^3 |v(t)|\,dt &= \int_0^{\frac{5}{3}} -v(t)\,dt + \int_{\frac{5}{3}}^3 v(t)\,dt \\
&= \int_0^{\frac{5}{3}} (5-3t)\,dt + \int_{\frac{5}{3}}^3 (3t-5)\,dt \\
&= \left[5t - \frac{3t^2}{2}\right]_{t=0}^{t=\frac{5}{3}} + \left[\frac{3t^2}{2} - 5t\right]_{t=\frac{5}{3}}^{t=3} \\
&= \left[5\left(\frac{5}{3}\right) - \frac{3}{2}\left(\frac{5}{3}\right)^2\right] - [0-0] + \left[\frac{3}{2}(3)^2 - 5(3)\right] - \left[\frac{3}{2}\left(\frac{5}{3}\right)^2 - 5\left(\frac{5}{3}\right)\right] \\
&= \frac{25}{3} - \frac{25}{6} + \frac{27}{2} - 15 - \frac{25}{6} + \frac{25}{3} \\
&= \frac{41}{6} \text{ meters}
\end{aligned}$$




**Takeaway** (For Net Displacement and Total Distance Traveled)**.**

$$\text{Net displacement on } [a,b] = \int_a^b v(t)\,dt,$$

$$\text{Total distance traveled on } [a,b] = \int_a^b |v(t)|\,dt,$$

where $|v(t)|$ should be expressed in a piecewise fashion. One should find the intervals on which $v(t)$ is positive and negative, respectively, and write

$$|v(t)| = \begin{cases} -v(t), & \text{the intervals on which } v(t) < 0 \\ v(t), & \text{the intervals on which } v(t) \geq 0 \end{cases}$$

and integrate the pieces over their corresponding intervals.

## 6.2 Function Symmetry

**Theorem 6.2** (Function Symmetry and Definite Integrals)**.** *For continuous even functions such that $f(-x) = f(x)$,*

$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx.$$

*For continuous odd functions such that $f(-x) = -f(x)$,*

$$\int_{-a}^{a} f(x)\, dx = 0.$$

*The definite integral on the left-hand side of each equation is known as **the definite integral of $f(x)$ over a symmetric interval about** $x = 0$.*

**Remark 6.1.** *Noticing symmetry is particularly useful in terms of saving time for calculations. Two steps are involved in using symmetry, and should be checked in this order:*

1. *The interval is symmetric about $x = 0$, e.g. $[-2, 2]$, but NOT $[-3, 4]$.*
2. *Verify the parity of the function.*

**Example 6.3** (Time saved for integrals of even functions)**.** Evaluate $\int_{-2}^{2} (3x^8 - 2)\, dx$.

**Solution.** We follow the steps of applying symmetry.

1. The interval is symmetric about $x = 0$. ✓
2. $f(-x) = 3(-x)^2 - 8 = 3x^2 - 8 = f(x)$, so $f(x)$ is even. ✓

Thus,

$$\begin{aligned}
\int_{-2}^{2} \left(3x^8 - 2\right) dx &\overset{\text{evenness}}{=} 2\int_{0}^{2} \left(3x^8 - 2\right) dx \\
&\overset{\text{FTC II}}{=} 2\left[\frac{1}{3}x^9 - 2x\right]_{x=0}^{x=2} \\
&= 2\left(\left[\frac{1}{3}2^9 - 2(2)\right] - [0 - 0]\right) \\
&= 2\left(\frac{512}{3} - 4\right) \\
&= \frac{1000}{3}
\end{aligned}$$

Can we verify the Theorem? Yes, just a little more brute force.

$$\begin{aligned}\int_{-2}^{2}\left(3x^8-2\right)dx &\overset{\text{FTC II}}{=} \left[\frac{1}{3}x^9-2x\right]_{x=-2}^{x=2} \\ &= \left[\frac{1}{3}2^9-2\left(2\right)\right]-\left[\frac{1}{3}\left(-2\right)^9-2\left(-2\right)\right] \\ &= \frac{512}{3}-4+\frac{512}{3}-4 \\ &= \frac{1024}{3}-8 \\ &= \frac{1000}{3}\end{aligned}$$

**Remark 6.2.** *The calculations saved are via the evaluation of $x = 0$ in the second term, which simply takes out half the algebra.*

**Example 6.4** (Time saved for integrals of odd functions)**.** Evaluate $\int_{-1}^{1}\left(x^3-x\right)dx$.

**Solution.** We follow the steps of applying symmetry.

1. The interval is symmetric about $x = 0$. ✓
2. $f\left(-x\right)=\left(-x\right)^3-\left(-x\right)=-x^3+x=-f\left(x\right)$, so $f\left(x\right)$ is odd. ✓

Thus,

$$\int_{-1}^{1}\left(x^3-x\right)dx=0.$$

Can we verify the Theorem? Yes, just a little more brute force.

$$\begin{aligned}\int_{-1}^{1}\left(x^3-x\right)dx &= \left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{x=-1}^{x=1} \\ &= \left[\frac{1}{4}-\frac{1}{2}\right]-\left[\frac{1}{4}-\frac{1}{2}\right] \\ &= \frac{1}{4}-\frac{1}{2}-\frac{1}{4}+\frac{1}{2} \\ &= 0\end{aligned}$$

**Remark 6.3.** *Notice that the time saved for odd functions integrated over a symmetric interval is immense.*

**Example 6.5** (Impossible Problem?)**.** Evaluate $\int_{-5}^{5}x^5e^{x^2}dx$.

**Solution.** We follow the steps of applying symmetry.

1. The interval is symmetric about $x = 0$. ✓
2. $f(-x) = (-x)^5 e^{(-x)^2} = -x^5 e^{x^2} = -f(x)$, so $f(x)$ is odd. ✓

Thus,

$$\int_{-5}^{5} x^5 e^{x^2} dx = 0$$

Can we verify the Theorem? This integral is impossible to do using the means we have learned so far. Regardless, we arrived at the correct answer using symmetry. There are **many** problems that can be solved using symmetry.

**Takeaway** (For Function Symmetry and Definite Integrals)**.**
The effort saved for an **odd** function is immense because one avoids evaluating a definite integral once for all.
The effort saved for an **even** function is half the algebra. However, one is still tasked with a "one-sided" integral.

**Fact 6.1** (Typical Even Functions)**.**

- $\cos(kx)$, *for any constant* $k$.
- $x^n$ *where* $n$ *is an even integer (including* $n = 0$*).*
- $e^{x^n}$ *where* $n$ *is an even integer (including* $n = 0$*).*
- *The product of two odd functions.*
- *The product of two even functions.*

**Fact 6.2** (Typical Odd Functions)**.**

- $\sin(kx)$, *for any constant* $k \neq 0$.
- $\tan(kx)$, *for any constant* $k \neq 0$.
- $x^n$ *where* $n$ *is an odd integer.*
- *The product of an even and odd function.*

**The motivation for the next section is**

After applying all the symmetry and tricks, what if we still have a difficult integral?

# 7 Lecture VII: Integration by Change of Variable

## 7.1 A Motivating Example

Consider the indefinite integral

$$\int \left(x^3 + x\right)^5 \left(3x^2 + 1\right) dx.$$

Um ... good luck finding the antiderivative quickly.

We should view the integrand as a product of two functions

$$f(x) = \left(x^3 + x\right)^5 \times \left(3x^2 + 1\right).$$

So, what kind of differentiation rule would possibly lead to a product? The Chain Rule.

Thus, the antiderivative must be written as a function composition,

$$F(x) = g(h(x)) \implies f(x) = F'(x) = g'(h(x)) \times h'(x).$$

So, we pair up our observations,

$$g'(h(x)) = \left(x^3 + x\right)^5,$$
$$h'(x) = 3x^2 + 1.$$

So, can we guess the function form of $g$? Yes, by looking at $h'(x) = 3x^2 + 1$, we find that

$$h(x) = x^3 + x + C.$$

We want $g'(h(x)) = (x^3 + x)^5$, so $C = 0$ and we identify

$$g'(x) = x^5 \implies g(x) = \frac{1}{6}x^6 + C.$$

Altogether,

$$F(x) = g(h(x)) = \frac{1}{6}\left(x^3 + x\right)^6 + C.$$

You can **verify** that this is the correct antiderivative by taking a derivative of $F(x)$, via the Chain Rule,

$$F'(x) = \frac{1}{6} \times 6\left(x^3 + x\right)^5 \times \left(3x^2 + 1\right) = \left(x^3 + x\right)^5 \left(3x^2 + 1\right),$$

which is precisely the original integrand.

## 7.2 Integration by Change of Variable

Now, let's put more structure into how to identify an antiderivative when we see things like this. We start with the chain rule itself. For a differentiable function $u(x)$, we have

$$\frac{d}{dx}\left(\frac{1}{n+1}(u(x))^{n+1}\right) = (u(x))^n \frac{du}{dx},$$

which implies

$$\frac{1}{n+1}(u(x))^{n+1} + C = \int u^n \frac{du}{dx} dx = \int u^n du.$$

Therefore, it is crucial to identify what $u$ is when $f(x)$ is presented in the above fashion. At the same time, we must check $du = \frac{du}{dx}dx = u'(x)\,dx$ and see if it is accompnied along. Here,

$$\int \left(x^3 + x\right)^5 \left(3x^2 + 1\right) dx,$$

we identify

$$u(x) = x^3 + x, \implies du = u'(x)\,dx = \left(3x^2 + 1\right) dx$$

using our knowledge of differentials. Thus,

$$\int \left(x^3 + x\right)^5 \left(3x^2 + 1\right) dx = \int u^5 du = \frac{u^6}{6} + C = \frac{\left(x^3 + x\right)^6}{6} + C.$$

**Example 7.1.** Evaluate $\int \sqrt{2x+1}dx$.

**Solution.** What is bothering you? Whatever is under that square root. Okay, it will go away if you set

$$u(x) = 2x + 1 \implies du = 2dx.$$

So, this suggests that the original problem becomes

$$\int \sqrt{2x+1}dx = \int \sqrt{u}dx$$

where we still cannot move forward because we are dealing with the wrong differential. We can find the correct differential via

$$du = 2dx \implies dx = \frac{du}{2}.$$

Together, we find

$$\int \sqrt{2x+1}dx = \int \sqrt{u}\frac{du}{2} = \frac{1}{2}\int u^{\frac{1}{2}} du = \frac{1}{2}\left(\frac{2}{3}u^{\frac{3}{2}}\right) + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C$$

**Theorem 7.1** (Change of Variable)**.** *If $u = g(x)$ is a differentiable function whose range is an interval $I$, and $f$ is continuous on $I$, then*

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

Consider the integral in the form $\int f(g(x))\, g'(x)\, dx$. First, identify what $f$ and $g$ are.

1. Substitute $u = g(x)$.
2. Write $du = g'(x)\, dx$ and construct $\int f(u)\, du$, namely, a variable change.
3. Integrate with respect to the new variable $u$.
4. After obtaining the antiderivative in $u$, replace $u$ with $g(x)$.

Sometimes, it is almost a matter of aesthetic. If you see a square root of an ugly function, then to reduce complexity, you should set the ugly function to be $u$, and hope everything works out afterwards. If it doesn't, then call something else $u$ and keep working.

## 7.3 Examples

**Example 7.2** (Trigonometric Functions)**.**

1. $$\int 3\sec^2(5x+1)\, dx.$$

   **Solution.** What's bad? Whatever is inside the secant. Set

   $$u = 5x + 1 \implies du = 5dx \implies dx = \frac{du}{5}.$$

   Thus,

   $$\begin{aligned}\int 3\sec^2(5x+1)\, dx &= 3\int \sec^2(u)\, \frac{du}{5} \\ &= \frac{3}{5}\int \sec^2(u)\, du \\ &= \frac{3}{5}\tan(u) + C \\ &= \frac{3}{5}\tan(5x+1) + C\end{aligned}$$

2.

$$\int \cos(7\theta + 3)\, d\theta.$$

**Solution.** What's ugly? Whatever is inside the cosine. Set

$$u = 7\theta + 3 \implies du = 7d\theta \implies d\theta = \frac{du}{7}.$$

Thus,

$$\int \cos(7\theta + 3)\, d\theta = \int \cos(u)\, \frac{du}{7} = \frac{1}{7} \int \cos(u)\, du = \frac{1}{7} \sin(u) + C = \frac{1}{7} \sin(7\theta + 3) + C$$

**Example 7.3** (Less Obvious Change of Variable)**.**

$$\int x^2 e^{x^3} dx.$$

**Solution.** What's bad? Whatever is inside the exponential. Set

$$u = x^3 \implies du = 3x^2 dx \implies x^2 dx = \frac{du}{3}.$$

Thus,

$$\begin{aligned}
\int x^2 e^{x^3} dx &= \int e^u \frac{du}{3} \\
&= \frac{1}{3} \int e^u du \\
&= \frac{1}{3} e^u + C \\
&= \frac{1}{3} e^{x^3} + C
\end{aligned}$$

# 8 Lecture VIII: Integrals Involving Exponential and Logarithmic Functions

## 8.1 More Examples on Indefinite Integrals using Change of Variable

**Example 8.1** (A simple example done properly)**.**

$$\int e^{-x}dx \overset{u=-x,\,du=-dx}{=} \int e^{u}\left(-du\right) = -\int e^{u}du = -e^{u}+C = -e^{-x}+C$$

**Example 8.2** (Another simple example done properly)**.**

$$\int \cos\left(3x\right)dx \overset{u=3x,\,du=3dx}{=} \int \cos\left(u\right)\frac{du}{3} = \frac{1}{3}\int \cos\left(u\right)du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin\left(3x\right)+C$$

**Example 8.3** (More to clean up)**.**

$$\int x\sqrt{2x+1}dx.$$

**Solution.** What's ugly? Whatever is inside the square root. Set

$$u = 2x+1 \implies du = 2dx \implies dx = \frac{du}{2}.$$

Thus,

$$\int x\sqrt{2x+1}dx = \int x\sqrt{u}\frac{du}{2}.$$

Still, we didn't quite rid of all the terms with $x$. But notice that

$$u = 2x+1 \implies x = \frac{u-1}{2}.$$

Thus,

$$\begin{aligned}
\int x\sqrt{2x+1}dx &\overset{u=2x+1}{=} \int x\sqrt{u}\frac{du}{2} \\
&\overset{x=\frac{u-1}{2}}{=} \frac{1}{2}\int \frac{u-1}{2}\sqrt{u}du \\
&= \frac{1}{4}\int (u-1)\, u^{\frac{1}{2}} du \\
&= \frac{1}{4}\int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du \\
&= \frac{1}{4}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right] + C \\
&= \frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + C.
\end{aligned}$$

This is NOT an obvious answer by just staring at the original function alone.

**Example 8.4** (Not so obvious Trigonometric Functions)**.**

$$\begin{aligned}
\int \tan(x)\, dx &= \int \frac{\sin(x)}{\cos(x)}dx \\
u=\cos(x), &\overset{du=-\sin(x)dx}{=} \int -\frac{1}{u}du \\
&= -\ln|u| + C \\
&= -\ln|\cos(x)| + C \\
&= \ln|\sec(x)| + C
\end{aligned}$$

where in the last step, we brought in the $-1$ as a power to $\cos(x)$, which is $\sec(x)$, due to the power extraction property of logarithm (email me if you don't know what this is).

Once again, you need to identify which function you should replace, and whether the derivative of it is in vision.

**Example 8.5** (Logarithmic Functions)**.**

$$\begin{aligned}
\int \frac{1}{x\ln x}dx \quad u=\ln x, &\overset{du=\frac{1}{x}dx}{=} \int \frac{1}{u}du \\
&= \ln|u| + C \\
&= \ln|\ln(x)| + C
\end{aligned}$$

**Takeaway.** Ugly functions should be substituted because if the ugly function goes away, the overall complexity is reduced, which is the precise goal of using a change of variable. The following cases are all **screaming** for a change of variable $u = g(x)$,

$$\sqrt{g(x)}, \frac{1}{g(x)}, (g(x))^n, \sin^n(g(x)), \cos^n(g(x)), \exp(g(x)), \ln(g(x)).$$

Though the list is not exhaustive, it is a good place to **start**. Only through a lot of practice problems can you gain experience and speed.

## 8.2 Change of Variable for Definite Integrals

We have seen a good number of indefinite integrals with the method of change of variable. This means that we are able to find the antiderivative even when the integrand looks a little nasty at the beginning. How does this method translate to **definite intgrals** where the bounds of integration are specified?

**Theorem 8.1** (Change of Variable for Definite Integrals)**.** *Let $u = g(x)$ and let $g'(x)$ be continuous over an interval $[a, b]$, and let $f$ be continuous over the range of $u = g(x)$. Then,*

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

*In other words, after applying the change of variable, one must also change the bounds of integration in the new variable.*

**Example 8.6** (Aperitivo)**.**

$$\int_0^{\pi/4} \sin(2x)\, dx$$

We propose the change of variable $u = 2x$ so $du = 2dx \implies dx = \frac{du}{2}$. At the same time, the bounds become

$$\begin{aligned} &\text{(lower bound)} \quad x = 0 \implies u = 0 \\ &\text{(upper bound)} \quad x = \frac{\pi}{4} \implies u = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2} \end{aligned}$$

Thus, altogether,

$$\begin{aligned}
\int_0^{\pi/4} \sin(2x)\, dx &= \int_0^{\frac{\pi}{2}} \sin(u)\, \frac{du}{2} \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin(u)\, du \\
&= \frac{1}{2} \left[-\cos(u)\right]_{u=0}^{u=\frac{\pi}{2}} \\
&= \frac{1}{2} \left[-\cos\left(\frac{\pi}{2}\right) - (-\cos(0))\right] \\
&= \frac{1}{2} [0+1] \\
&= \frac{1}{2}
\end{aligned}$$

**Example 8.7** (Antipasto)**.**

$$\int_0^1 2xe^{4x^2+3} dx$$

Set $u = 4x^2 + 3$. Then $du = 8xdx \implies 2xdx = \frac{du}{4}$. Now, the bounds change to

$$\begin{aligned}
&\text{(lower bound)} \quad x = 0 \implies u = 4(0)^2 + 3 = 3 \\
&\text{(upper bound)} \quad x = 1 \implies u = 4(1)^2 + 3 = 7
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^1 2xe^{4x^2+3} dx &= \int_3^7 e^u \frac{du}{4} \\
&= \frac{1}{4} \int_3^7 e^u du \\
&= \frac{1}{4} \left[e^u\right]_{u=3}^{u=7} \\
&= \frac{1}{4} \left[e^7 - e^3\right]
\end{aligned}$$

**Example 8.8** (Primo Piatto)**.**

$$\int_{-1}^0 y\left(2y^2 - 3\right)^5 dy$$

Let

$$u = 2y^2 - 3 \implies du = 4ydy \implies ydy = \frac{du}{4}.$$

Furthermore, the bounds change to

$$\begin{aligned}(\text{lower bound}) \quad y = -1 &\implies u = 2\,(-1)^2 - 3 = -1 \\ (\text{upper bound}) \quad y = 0 &\implies u = 2\,(0)^2 - 3 = -3\end{aligned}$$

Therefore, combining both the change of variable and the integration bounds update, we have

$$\begin{aligned}\int_{-1}^{0} y\left(2y^2-3\right)^5 dy &= \int_{-1}^{-3} u^5 \frac{du}{4} \\ &= \frac{1}{4}\int_{-1}^{-3} u^5 du \\ &= \frac{1}{4}\left[\frac{1}{6}u^6\right]_{u=-1}^{u=-3} \\ &= \frac{1}{24}\left[(-3)^6 - (-1)^6\right] \\ &= \frac{1}{24}\left[729-1\right] \\ &= \frac{91}{3}\end{aligned}$$

Note that it is perfectly fine to have a larger lower bound after performing the change of variable. Trust the math when it is nice and clear, over the cognitive inertia that "stars must align properly".

**Example 8.9** (Secondo Piatto)**.**

$$\int_0^{\frac{\pi}{2}} \cos^2(\theta)\, d\theta$$

We have nowhere to make a change of variable. One may suggest that $u = \cos(\theta)$ so the integrand may reduce to just $u^2$. But the price to pay is the differential $du = -\sin\theta d\theta$, which is nowhere to be found in the integrand. Therefore, something smarter must be done.

The core issue lies in the higher power of the cosine. If this problem can be reduced in some way to the one in Aperitivo (see Example 8.6), then we are set. Luckily, there is such a trigonometric identity

$$\cos^2(\theta) = \frac{1+\cos 2\theta}{2}.$$

Thus,

$$\int_0^{\frac{\pi}{2}} \cos^2(\theta)\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\, d\theta.$$

Now, for those who can integrate using the reverse chain rule very quickly, be my guest and proceed more quickly. For those who want to see the change of variable, follow me.

Set $u = 2\theta$, and thus $du = 2d\theta \implies d\theta = \frac{1}{2}du$. Furthermore, the new bounds in $u$ become

$$\begin{aligned} &\text{(lower bound)} \quad \theta = 0 \implies u = 2\,(0) = 0 \\ &\text{(upper bound)} \quad \theta = \frac{\pi}{2} \implies u = 2\left(\frac{\pi}{2}\right) = \pi \end{aligned}$$

Thus,

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \cos^2(\theta)\, d\theta &= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)\, d\theta \\ &= \frac{1}{2}\int_0^{\pi} (1 + \cos u)\, \frac{du}{2} \\ &= \frac{1}{4}\int_0^{\pi} (1 + \cos u)\, du \\ &= \frac{1}{4}\left[u + \sin u\right]_{u=0}^{u=\pi} \\ &= \frac{1}{4}\left[\left(\pi + \cancelto{0}{\sin \pi}\right) - (0 + 0)\right] \\ &= \frac{1}{4}\pi \end{aligned}$$

# 9 Lecture IX: Integration by parts

## 9.1 A Motivating Example

In the last homework, we were tasked to find the indefinite integral

$$\int x \ln x dx, \quad x\text{-problem},$$

which we call the "$x$-problem". The trick was to rewrite

$$\ln x = \frac{1}{2} \ln x^2,$$

and use a change of variable $u = x^2$. This turns the indefinite integral into

$$\int x \ln x dx = \frac{1}{2} \int x \ln x^2 dx = \frac{1}{4} \int \ln u du, \quad u\text{-problem},$$

which we call the "$u$-problem".

Then, we quoted directly a formula provided by the textbook

$$\begin{aligned} \int \ln u dx &= u\left[\ln u - 1\right] + C \\ \implies \int x \ln x dx &= \frac{1}{4} \int \ln u du \\ &= \frac{1}{4} x^2 \left[\ln x^2 - 1\right] + C \\ &= \frac{1}{4} x^2 \left[2 \ln x - 1\right] + C \\ &= \frac{x^2}{2} \ln x - \frac{1}{4} x^2 + C \end{aligned}$$

**This demands a proper justification.**

Now, knowing that the $u$-problem and the $x$-problem are equivalent, it doesn't hurt to study *just* the $x$-problem. We were prompted to use the method of change of variable when we observe a product of two functions, as presented in the $x$-problem. But unluckily, setting $u = \ln x$ gives us

$$\int x \ln x dx = \int u e^{2u} du$$

since $u = \ln x, du = \frac{1}{x} dx \implies xdx = x^2 du = e^{2u} du$. This is still a product of two functions, which means we didn't quite reduce the complexity.

So, what other differentiation rule is associated with a product of two functions? The chain rule, which inspired the method of change of variable, is indeed one. But the product rule also involves a product of two function.

$$\frac{d}{dx}\left(f\left(x\right)g\left(x\right)\right) = f'\left(x\right)g\left(x\right) + f\left(x\right)g'\left(x\right)$$

Now, if we integrate both sides against $x$, via the FTC I, we have

$$\begin{aligned} \int \frac{d}{dx}\left(f\left(x\right)g\left(x\right)\right)dx &= \int f'\left(x\right)g\left(x\right)dx + \int f\left(x\right)g'\left(x\right)dx \\ \implies f\left(x\right)g\left(x\right) &= \int f'\left(x\right)g\left(x\right)dx + \int f\left(x\right)g'\left(x\right)dx \end{aligned}$$

Rearranging, we obtain

$$\int f\left(x\right)g'\left(x\right)dx = f\left(x\right)g\left(x\right) - \int f'\left(x\right)g\left(x\right)dx. \tag{3}$$

Thus, by making a change of variable

$$\begin{aligned} u = f\left(x\right) &\implies du = f'\left(x\right)dx \\ v = g\left(x\right) &\implies dv = g'\left(x\right)dx, \end{aligned}$$

we can rewrite Equation (3) as

$$\int udv = uv - \int vdu \tag{4}$$

which is the celebrated, **Integration by parts** formula.

## 9.2 Integration-by-parts Formula

**Theorem 9.1** (Integration-by-parts)**.** *Let $u = f\left(x\right)$ and $v = g\left(x\right)$ be functions with continuous derivatives. Then, the integration-by-parts formula for the integral involving these two functions is:*

$$\int udv = uv - \int vdu. \tag{5}$$

**Remark 9.1.** *The advantage of using the integration-by-parts formula is that we can use it to exchange one integral for another, possibly easier, integral. In other words, you are transferring the difficulty of one integral to another.*

**Example 9.1** (Simple but effective)**.** Evaluate $\int x \sin x dx$.

**Discussion 9.1.** One general observation is to see which function can be simplified if differentiated, and which function stays about the same if integrated. Set the former as $u$, and the latter as $dv$. In this case, we see that the function $u = x$ goes away by differentiating, while $dv = \sin x dx$ stays trigonometric if integrated.

**Solution.** Thus, via this change of variable, we identify

$$\int x \sin(x)\, dx = \int u dv = uv - \int v du. \tag{6}$$

We formulate the table

$$\begin{aligned} u = x, &\quad du = dx \\ dv = \sin(x)\, dx, &\quad v = -\cos(x) \end{aligned}$$

Using Equation (6), we see that

$$\int x \sin(x)\, dx = \int u dv = uv - \int v du = -x\cos(x) - \int -\cos(x)\, dx = -x\cos(x) + \sin(x) + C$$

**Question 9.1.** What if you do $u = \sin x$ and $dv = x dx$? Try it at home. There is nothing wrong with this choice in the sense that it will produce the same answer, but with much longer algebra.

**Example 9.2** (Old problem revisited, with justifications)**.**

$$\int x \ln x dx,$$

where we use

$$\begin{aligned} u = \ln(x), &\quad du = \frac{1}{x} dx \\ dv = x dx, &\quad v = \frac{x^2}{2}. \end{aligned}$$

Thus,

$$\int x \ln(x)\, dx = \int u dv = uv - \int v du = x^2 \ln(x) - \int \frac{x^2}{2} \frac{1}{x} dx = \frac{x^2}{2} \ln(x) - \frac{1}{4} x^2 + C,$$

as expected.

**Takeaway 9.1.** Your job: identify $u$ and $dv$. And it might take you several trials.

## 9.3 LIATE

Trial-and-error, as we know from experience, works but is time-consuming. Is there a better "tell" to choose $u$ and $dv$? We employ the following mnemonic.

$$\xrightarrow{\text{Simplifying: Decreasing complexity when differentiated}}$$

$$u \cdots \textbf{L}\text{ogarithmic} > \textbf{I}\text{nverse Trig} > \textbf{A}\text{lgebraic} > \textbf{T}\text{rigonometric} > \textbf{E}\text{xponential} \cdots dv$$

$$\xleftarrow[\text{Price to pay: Increasing complexity when integrated}]{}$$

This is the hierarchy we abide when doing integration by parts. For example, when given a product of a logarithmic function $L(x)$ and an algebraic function $A(x)$, we always set $u = L(x)$, and $dv = A(x)\,dx$. The reason for this is that the logarithmic part, after differentiation, becomes significantly simpler. The price to pay, when integrating $A(x)$, is not so bad.

**Example 9.3.** Evaluate $\int xe^{3x}dx$.

**Solution.** By the mnemonic, we select

$$\begin{aligned} u &= x, \quad du = dx \\ dv &= e^{3x}dx, \quad v = \frac{1}{3}e^{3x}. \end{aligned}$$

Thus,

$$\int xe^{3x}dx = \int udv = uv - \int vdu = \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C.$$

**Example 9.4** (Integration-by-parts, Twice)**.** Evaluate $\int x^2e^{3x}dx$.

**Solution.** Even though this problem looks similar to the last one, which implies a similar procedure, let's focus on the slight differences and see if we can develop some intuition.

By the mnemonic, we select

$$\begin{aligned} u &= x^2, \quad du = 2x^dx \\ dv &= e^{3x}dx, \quad v = \frac{1}{3}e^{3x}. \end{aligned}$$

Thus, using the result of the last problem,

$$\int x^2e^{3x}dx = \int udv = uv - \int vdu = \frac{1}{3}x^2e^{3x} - \frac{2}{3}\int xe^{3x}dx = \frac{1}{3}xe^{3x} - \frac{2}{3}\left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right] + C$$

**Question 9.2** (Bonus: see HW3)**.** Find a general formula for

$$\int x^n e^x dx$$

for any integer $n \neq -1$.

# 10 Lecture X: Integration by parts, Definite Integrals and Examples

We continue from last class but now work on examples where the LIATE principle no longer works.

## 10.1 When LIATE doesn't work

**Example 10.1.** Evaluate

$$\int t^3 e^{t^2} dt$$

**Solution.** If we follow LIATE, we will have

$$\begin{aligned} u &= t^3, \quad du = 3t^2 dt \\ dv &= e^{t^2} dt, \quad v = ??? \end{aligned}$$

where we run into the issue of not knowing the antiderivative of $e^{t^2}$.

However, we do know how to integrate $\int te^{t^2} dt$ via substitution. So, we should put ourselves in that position. To achieve this, we use

$$\begin{aligned} u &= t^2, \quad du = 2tdt, \\ dv &= te^{t^2} dt, \quad v = \frac{1}{2}e^{t^2}. \end{aligned}$$

Thus,

$$\begin{aligned} u &= t^2, \quad du = 2tdt, \\ dv &= te^{t^2} dt, \quad v = \frac{1}{2}e^{t^2}. \end{aligned}$$

We follow what we have identified and write

$$\begin{aligned} \int t^3 e^{t^2} dt &= \int \left(t^2\right)\left(te^{t^2} dt\right) = \int udv = uv - \int vdu \\ &= \frac{t^2}{2}e^{t^2} - \int \left(\frac{1}{2}e^{t^2}\right)(2tdt) = \frac{t^2}{2}e^{t^2} - \int te^{t^2} dt = \frac{t^2}{2}e^{t^2} - \frac{1}{2}e^{t^2} + C \end{aligned}$$

## 10.2 When there is only one integrand

Let's find the antiderivative of $\ln x$. This was the "$u$-problem" in Section 9. We are now able to **justify** the formula for the antiderivative of natural log.

**Theorem 10.1** (Antiderivative of natural log)**.**

$$\int \ln(x)\,dx = x\left(\ln(x) - 1\right) + C.$$

*Proof.* We integrate by parts using

$$\begin{aligned} u &= \ln(x)\,, \quad du = \frac{1}{x}dx \\ dv &= dx, \quad v = x \end{aligned}$$

which highlights a neat trick of using the function $dv = 1dx$.

Thus,

$$\int \ln(x)\,dx = \int \left(\ln(x)\right)(1dx) = \int udv = uv - \int vdu$$
$$= x\ln(x) - \int x\frac{1}{x}dx = x\ln(x) - \int dx = x\ln(x) - x + C = x\left(\ln(x) - 1\right) + C$$

□

## 10.3 Integration-by-parts More Than Once

**Example 10.2.** Evaluate $\int \sin \ln x dx$.

**Solution.** Just like the one with $\ln x$, where we only observe one function in the integrand, we use $dv = 1dx$ as the other function.

Set

$$\begin{aligned} u &= \sin(\ln x)\,, \quad du = \frac{\cos(\ln x)}{x}dx; \\ dv &= dx, \quad v = x. \end{aligned}$$

Thus,

$$\begin{aligned} \int \sin(\ln x)\,dx &= uv - \int vdu \\ &= x\sin(\ln x) - \int x\frac{\cos(\ln x)}{x}dx \\ &= x\sin(\ln x) - \int \cos(\ln x)\,dx \end{aligned}$$

Now, we need to deal with the second integral, which is equally difficult. However, do not give up. We integrate by parts again. Set

$$\begin{aligned} u = \cos(\ln x), \quad du &= -\frac{\sin(\ln x)}{x} dx \\ dv = dx, \quad v &= x \end{aligned}$$

Thus,

$$\begin{aligned} \int \cos(\ln x)\, dx &= uv - \int v du \\ &= x\cos(\ln x) - \int x\left(-\frac{\sin(\ln x)}{x}\right) dx \\ &= x\cos(\ln x) + \int x\sin(\ln x)\, dx \end{aligned}$$

Now, define $I = \int \sin(\ln x)\, dx$, the original integral. Piecing the above information together, we have

$$\begin{aligned} I = \int \sin(\ln x)\, dx &= x\sin(\ln x) - \int \cos(\ln x)\, dx \\ &= x\sin(\ln x) - x\cos(\ln x) - I + C \end{aligned}$$

In other words,

$$I = x\sin(\ln x) - x\cos(\ln x) - I + C \overset{\text{add } I \text{ on both sides}}{\Longrightarrow} I = \frac{x\sin(\ln x) - x\cos(\ln x)}{2} + C$$

**Discussion 10.1.** Come up with your own similar example and solve it. You may try Exercise 47 in Section 3.1 in OpenStax Vol. 2, first.

## 10.4 Definite Integral

This is a straightforward procedure.

**Theorem 10.2** (Integration-by-parts for Definite Integrals)**.** *Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives on $[a, b]$. Then,*

$$\int_a^b u dv = [uv]_{x=a}^{x=b} - \int_a^b v du$$

**Example 10.3.** Evaluate $\int_0^1 xe^{-2x} dx$.

**Solution.** By LIATE, we select

$$u = x, \quad du = dx;$$
$$dv = e^{-2x}dx, \quad v = -\frac{1}{2}e^{-2x}.$$

Then, following integration-by-parts, we have

$$\begin{aligned}
\int_0^1 xe^{-2x}dx &= [uv]_{x=0}^{x=1} - \int_0^1 vdu \\
&= \left[-\frac{x}{2}e^{-2x}\right]_{x=0}^{x=1} - \int_0^1 -\frac{1}{2}e^{-2x}dx \\
&= \left[-\frac{1}{2}e^{-2} - \left(-\frac{0}{2}e^0\right)\right] + \frac{1}{2}\int_0^1 e^{-2x}dx \\
&= -\frac{1}{2}e^{-2} + \frac{1}{2}\left[-\frac{1}{2}e^{-2x}\right]_{x=0}^{x=1} \\
&= -\frac{1}{2}e^{-2} + \frac{1}{2}\left[-\frac{1}{2}e^{-2} - \left(-\frac{1}{2}e^0\right)\right] \\
&= -\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2} + \frac{1}{4} \\
&= -\frac{3}{4}e^{-2} + \frac{1}{4}
\end{aligned}$$

Moral of the story: consistent book-keeping gets the job done.

**Example 10.4.** Evaluate $\int_0^1 (\ln x^2 + 1)\, dx$.

**Solution.** By LIATE, we select

$$u = \ln\left(x^2+1\right), \quad du = \frac{2x}{x^2+1}dx$$
$$dv = dx, \quad v = x$$

By integration-by-parts, we have

$$\begin{aligned}
\int_0^1 \ln\left(x^2+1\right) dx &= \left[x\ln\left(x^2+1\right)\right]_{x=0}^{x=1} - \int_0^1 x\frac{2x}{x^2+1}dx \\
&= [\ln(2) - 0] - 2\boxed{\int_0^1 \frac{x^2}{x^2+1}dx} \\
&= \ln(2) - 2\left(1 - \frac{\pi}{4}\right) \\
&= \ln 2 - 2 + \frac{\pi}{2}
\end{aligned}$$

where we explicitly compute the integral in the box

$$\begin{aligned}
\int_0^1 \frac{x^2}{x^2+1}dx &= \int_0^1 \frac{x^2+1-1}{x^2+1}dx \\
&= \int_0^1 1dx - \int_0^1 \frac{1}{x^2+1}dx \\
&= [x]_{x=0}^{x=1} - \left[\tan^{-1}(x)\right]_{x=0}^{x=1} \\
&= 1 - \left[\tan^{-1}(1) - \tan^{-1}(0)\right] \\
&= 1 - \frac{\pi}{4}
\end{aligned}$$

# 11 Lecture XI: Integrals of Trigonometric Functions

In the last few lectures, we have seen an awful dosage of integrals such as

$$\int \cos x dx, \quad \int \sin 2x dx, \quad \int \tan x dx,$$

or even ones a bit more nasty such as

$$\int \cos^2(x)\, dx, \quad \int \sin^2(x)\, dx.$$

The first set is not terrible, but the second set requires that we know some trigonometric identities. The goal, at the end of the day, is to reduce the problem in the second set to the first, via any mathematical means. Conscious practices of learning which type of identity one must use (or whether we need one at all) is the key to success in integration of trigonometric functions.

The identities we often call for are

$$\begin{aligned} \sin^2(x) + \cos^2(x) &= 1 \\ \tan^2(x) + 1 &= \sec^2(x) \\ \cos^2(x) &= \frac{1+\cos(2\theta)}{2} \\ \sin^2(x) &= \frac{1-\cos(2\theta)}{2} \end{aligned}$$

## 11.1 Sine and Cosine: Odd Powers

Let's start with a simpler teaser problem.

**Example 11.1.** For any integer $n$, evaluate the integral

$$\int \cos^n(x)\sin(x)\, dx$$

**Solution.** The $\sin(x)$ here is the angel to help us. Using the change of variable,

$$u = \cos(x) \implies du = -\sin(x)\, dx.$$

we have

$$\begin{aligned} \int \cos^n(x)\sin(x)\, dx \quad {}^{u=\cos(x),} \quad &\overset{du=-\sin(x)dx}{=} \quad -\int u^n du \\ &= \quad -\frac{u^{n+1}}{n+1} + C \\ &= \quad -\frac{\cos^{n+1}(x)}{n+1} + C \end{aligned}$$

A similar technique can be applied to

$$\int \sin^n(x)\cos(x)\,dx.$$

In other words, if we are in either form, we are in good shape because a simple change of variable takes us directly to the antiderivative. So, the goal of the following problems is to put ourselves in such a form.

Now, let's face a harder problem.

**Example 11.2.** Evaluate

$$\int \cos^2(x)\sin^3(x)\,dx$$

**Solution.** So, we are not in the two forms as described earlier. Some work must be done.

We wish the cubic of sine is just a single power, while the rest of the expression can be written entirely in terms of cosine so a change of variable of $u = \cos x$ gets the job done. The famous trigonometric identity comes to the rescue.

Note that

$$\sin^3(x) = \sin^2(x)\sin(x) = \left(1-\cos^2(x)\right)\sin(x).$$

This turns the original integral to

$$\begin{aligned}
\int \cos^2(x)\sin^3(x)\,dx &= \int \cos^2(x)\sin^2(x)\sin(x)\,dx \\
&= \int \cos^2(x)\left(1-\cos^2(x)\right)\sin(x)\,dx \\
&\overset{u=\cos(x),\quad du=-\sin(x)dx}{=} -\int u^2\left(1-u^2\right)du \\
&= \int \left(u^4-u^2\right)du \\
&= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\
&= \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + C
\end{aligned}$$

The problem

$$\int \cos^3(x)\sin^2(x)\,dx$$

can be solved in a similar fashion where we kill the odd cosine power instead, using the same identity as the last example.

**Question 11.1** (Bonus problem)**.** So, can we solve the more general problem? Evaluate

$$\int \cos^n(x)\sin^m(x)\,dx$$

where $m$ is an odd integer, and $n$ is any integer.

Note that integrals such as

$$\int \cos^3(x)\,dx, \quad \int \sin^3(x)\,dx$$

are special cases of the general problem.

**Example 11.3** (We love odd powers)**.**

$$\begin{aligned}
\int \cos^3(x)\,dx &= \int \cos(x)\cos^2(x)\,dx \\
&= \int \cos(x)\left(1-\sin^2(x)\right)dx \\
&\overset{u=\sin x,\quad du=\cos(x)dx}{=} \int \left(1-u^2\right)du \\
&= u - \frac{u^3}{3} + C \\
&= \sin(x) - \frac{\sin^3(x)}{3} + C
\end{aligned}$$

**Question 11.2.** Can you solve

$$\int \sin^5(x)\,dx?$$

## 11.2 Sine and Cosine: Even Powers

In the last section, we saw examples where the integrand always contains odd powers of either sine or cosine, which forms a derivative/antiderivative relationship conveniently. Their integrals tend to be more straightforward with a direct change of variable, after some manipulations.

The cases we now need to deal with is just an integrand with even trigonometric powers. Instead of a direct change of variable, we make use of the double angle formula.

**Example 11.4** (Squares)**.**

$$\begin{aligned}\int \cos^2(x)\,dx &= \int \frac{1+\cos(2x)}{2}dx\\ &= \frac{1}{2}\int (1+\cos(2x))\,dx\\ &= \frac{1}{2}\left[x+\frac{\sin 2x}{2}\right]+C\end{aligned}$$

**Example 11.5** (Squares cont.)**.** Note that the same technique can be applied to problems such as

$$\begin{aligned}\int \sin^2(3x)\,dx &= \int \frac{1-\cos(6x)}{2}dx\\ &= \frac{1}{2}\int (1-\cos(6x))\,dx\\ &= \frac{1}{2}\left(x-\frac{\sin(6x)}{6}\right)+C\end{aligned}$$

where you pay extra attention to the **double angle formula**, i.e., the angle is **doubled** from $3x$ to $6x$.

What if both sine and cosine show up in even powers?

**Example 11.6** (Double angle formula applied twice)**.**

$$\begin{aligned}\int \cos^2(x)\sin^2(x)\,dx &\overset{\text{double angle formula}}{=} \int \left(\frac{1+\cos(2x)}{2}\right)\left(\frac{1-\cos(2x)}{2}\right)dx\\ &= \frac{1}{4}\int (1+\cos(2x))(1-\cos(2x))\,dx\\ &= \frac{1}{4}\int \left(1-\cos^2(2x)\right)dx\\ &\overset{\text{double angle formula}}{=} \frac{1}{4}\int \left(1-\frac{1+\cos(4x)}{2}\right)dx\\ &= \frac{1}{4}\int \left(1-\frac{1}{2}-\frac{\cos(4x)}{2}\right)dx\\ &= \frac{1}{4}\int \left(\frac{1}{2}-\frac{\cos(4x)}{2}\right)dx\\ &= \frac{1}{8}\int (1-\cos(4x))\,dx\\ &= \frac{1}{8}\left(x-\frac{\sin(4x)}{4}\right)+C\end{aligned}$$

where a lot of algebraic brooming is done to show all details.

**Question 11.3.** Can you solve

$$\int \cos^4(x)\sin^4(x)\,dx?$$

What about

$$\int \cos^6(x)\sin^6(x)\,dx?$$

They both use the same technique at first and then branch off. What's the difference? If you can answer this, you don't need to read the next section.

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## 11.3 Overall Strategy (A Cautious Read)

So, it seems we have an awful lots of varieties here. How do we practice to make our success rate higher?
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**Strategy 11.1** (Sine and Cosine)**.**
Problem:

$$\int \cos^m(x)\sin^n(x)\,dx.$$

Weapons: identity $\sin^2(x)+\cos^2(x)=1$ and double angle formula,

$$\cos^2(x) = \frac{1+\cos(2\theta)}{2}$$
$$\sin^2(x) = \frac{1-\cos(2\theta)}{2}$$

1. If $n$ is odd, then we write

$$\sin^n(x)=\sin^{n-1}(x)\sin(x)$$

**in preparation to use a change of variable** $u=\cos x$. $\sin^{n-1}(x)$ now has an even power, which can then be fully converted to an expression of $\cos x$ using the identity $\sin^2(x)=1-\cos^2(x)$.

2. If $m$ is odd, then we isolate

$$\cos^n(x) = \cos^{n-1}(x)\cos(x)$$

   **in preparation to use a change of variable** $u = \sin x$. $\cos^{n-1}(x)$ now has an even power, which can then be fully converted to an expression of $\sin x$ using the identity $\sin^2(x) = 1 - \cos^2(x)$.

3. If both $m$ and $n$ are even, then use the double angle formula to reduce the problem into either case 1 or 2 and apply the method there.

# 12 Lecture XII: Integrals of Trigonometric Functions cont.

## 12.1 Tangent and Secant

Now that we have dealt with sines and cosines with the help of trigonometric identities involving them, it's time to face their evil twin: tangent and secant. We are luckily blessed with a nice trigonometric identity for them

$$\tan^2(x) + 1 = \sec^2(x). \tag{7}$$

Can't remember this? There is no excuse because you can derive it on the spot.

$$\begin{aligned} \sin^2(x) + \cos^2(x) &= 1 \\ \overset{\text{divide both sides by } \cos^2(x)}{\Longrightarrow} \frac{\sin^2(x)}{\cos^2(x)} + 1 &= \frac{1}{\cos^2(x)} \\ \Longrightarrow \tan^2(x) + 1 &= \sec^2(x) \end{aligned}$$

There are a few more basic integrals you should know by heart now.

$$\begin{aligned} \int \sec^2(x)\,dx &= \tan(x) + C; \\ \int \sec(x)\tan(x)\,dx &= \sec(x) + C; \\ \int \tan(x)\,dx &= \ln|\sec(x)| + C; \text{(see } \textit{Example } 8.4) \\ \int \sec(x)\,dx &= \ln|\sec(x) + \tan(x)| + C. \end{aligned}$$

The last one seems a bit obscure. We prove it here.

**Theorem 12.1** (Indefinite Integral of Secant)**.**

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$$

*Proof.*

$$\begin{aligned}
\int \sec(x)\,dx &\overset{\text{the least obvious step}}{=} \int \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}dx \\
&= \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\tan(x)+\sec(x)}dx \\
&\underset{du=(\sec^2(x)+\sec(x)\tan(x))dx}{\overset{u=\tan(x)+\sec(x)}{=}} = \int \frac{1}{u}du \\
&= \ln|u|+C \\
&= \ln|\tan(x)+\sec(x)|+C
\end{aligned}$$

□

Equipped with the basic integrals and the identity per Equation (7), we can solve many problems involving integrals of tangent and secant. The technique revolves around your familiarity of the basic information.

**Example 12.1.** Evaluate

$$\int \sec^5(x)\tan(x)\,dx$$

**Solution.**

$$\begin{aligned}
\int \sec^5(x)\tan(x)\,dx &= \int \sec^4(x)\left(\sec(x)\tan(x)\,dx\right) \\
&\underset{du=\sec(x)\tan(x)dx}{\overset{u=\sec(x)}{=}} \int u^4du \\
&= \frac{1}{5}u^5+C \\
&= \frac{1}{5}\sec^5(x)+C
\end{aligned}$$

**Takeaway.** A lone $\tan(x)$ is not very helpful, either as a derivative or as an integral. It becomes more convenient when paired with $\tan(x)\sec(x)$ so that it can be used as the differential of $\sec(x)$.

With this in mind, let's try another.

**Example 12.2.** Evaluate

$$\int \sec^3(x)\tan^3(x)\,dx$$

**Solution.**

$$\begin{aligned}
\int \sec^3(x)\tan^3(x)\,dx &= \int \sec^2(x)\tan^2(x)\left(\sec(x)\tan(x)\,dx\right)\\
&= \int \sec^2(x)\left(\sec^2(x)-1\right)\left(\sec(x)\tan(x)\,dx\right)\\
&\underset{du=\sec(x)\tan(x)}{\overset{u=\sec(x)}{=}} \int u^2\left(u^2-1\right)du\\
&= \int \left(u^4-u^2\right)du\\
&= \frac{1}{5}u^5-\frac{1}{3}u^3+C\\
&= \frac{1}{5}\sec^5(x)-\frac{1}{3}\sec^3(x)+C
\end{aligned}$$

**Takeaway.** In this example, we not only split out the factor of $\sec(x)\tan(x)$ to form a differential, but also used the identity to reduce the remaining $\tan^2(x)$ into a quantity involving $\sec(x)$, which is what we wanted as the change of variable in the first place (by forming the corresponding differential).

Note that in the last two examples, the power of secant is always odd. What if it is even?

**Example 12.3** (Square powers of Secant)**.**

$$\begin{aligned}
\int \tan^5(x)\sec^2(x)\,dx &\underset{du=\sec^2(x)dx}{\overset{u=\tan(x)}{=}} \int u^5du\\
&= \frac{1}{6}u^6+C\\
&= \frac{1}{6}\tan^6(x)+C
\end{aligned}$$

which isn't that terrible.

What if the power is even but more than 2?

**Example 12.4** (Higher even powers of Secant)**.**

$$
\begin{aligned}
\int \tan^5(x)\sec^4(x)\,dx &= \int \tan^5(x)\sec^2(x)\sec^2(x)\,dx \\
&\overset{\tan^2(x)+1=\sec^2(x)}{=} \int \tan^5(x)\left(\tan^2(x)+1\right)\sec^2(x)\,dx \\
&\underset{du=\sec^2(x)dx}{\overset{u=\tan(x)}{=}} \int u^5\left(u^2+1\right)du \\
&= \int \left(u^7+u^5\right)du \\
&= \frac{1}{8}u^8+\frac{1}{6}u^6+C \\
&= \frac{1}{8}\tan^8(x)+\frac{1}{6}\tan^6(x)+C
\end{aligned}
$$

**Takeaway.** With even powers of secant, we can always extract a differential $\sec^2(x)$ which means we want to make a change of variable $u=\tan(x)$. The remaining even powers of secant can be further reduced to an expression involving $\tan^2(x)$ using the identity.

In the case for secant, we actually love the fact that secant has an **even** power because it automatically lends itself to the differential of $\tan x$.

How about tangent or secant just by themselves?

**Example 12.5** (Tangent cubed)**.** Evaluate

$$\int \tan^3(x)\,dx.$$

**Solution.**

$$
\begin{aligned}
\int \tan^3(x)\,dx &= \int \tan(x)\tan^2(x)\,dx \\
&= \int \tan(x)\left(\sec^2(x)-1\right)dx \\
&= \int \tan(x)\sec^2(x)\,dx - \int \tan(x)\,dx \\
&\overset{u=\tan(x)}{=} \int u du - \ln|\sec(x)| \\
&= \frac{1}{2}u^2 - \ln|\sec(x)| + C \\
&= \frac{1}{2}\tan^2(x) - \ln|\sec(x)| + C
\end{aligned}
$$

Another infamous example is just an odd power of secant. So far, we have used exclusively the method of change of variable. Now, it's time for integration by parts to shine.

**Example 12.6** (Secant cubed)**.** Evaluate

$$\int \sec^3(x)\,dx.$$

**Solution.** Via integration-by-parts using

$$\begin{aligned} u &= \sec(x), \quad & du &= \sec(x)\tan(x)\,dx \\ dv &= \sec^2(x)\,dx, \quad & v &= \tan(x) \end{aligned}$$

we have

$$\begin{aligned} \int \sec^3(x)\,dx &= \int \underbrace{\sec(x)}_{u}\overbrace{\sec^2(x)\,dx}^{dv} \\ &= \underbrace{\sec(x)\tan(x)}_{uv} - \int \underbrace{\sec(x)\tan^2(x)\,dx}_{vdu} \\ &= \sec(x)\tan(x) - \int \sec(x)\left(\sec^2(x) - 1\right)dx \\ &= \sec(x)\tan(x) - \int \left(\sec^3(x) - \sec(x)\right)dx \\ &= \sec(x)\tan(x) - \int \sec^3(x)\,dx + \int \sec(x)\,dx. \end{aligned}$$

Rearranging by adding $\int \sec^3(x)\,dx$ to both sides, we have

$$\begin{aligned} 2\int \sec^3(x)\,dx &= \sec(x)\tan(x) + \int \sec(x)\,dx \\ &= \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| + C \\ \implies \int \sec^3(x)\,dx &= \frac{\sec(x)\tan(x)}{2} + \frac{1}{2}\ln|\sec(x) + \tan(x)| + C \end{aligned}$$

DO NOT READ NEXT IF YOU HAVEN'T DONE ANY EXERCISES!!!!!!!!!!!!!!!!!!!!!
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## 12.2 Overall Strategy (A Cautious Read)

So, it seems we have an awful lots of varieties here. How do we practice to make our success rate higher?

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**Strategy 12.1** (Tangent and Secant)**.**

Problem:

$$\int \tan^m(x)\sec^n(x)\,dx.$$

Weapons: identity $\tan^2(x) + 1 = \sec^2(x)$ and the derivatives

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$
$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

1. If $n \geq 2$ is even, we extract a secant square

   $$\sec^n(x) = \sec^{n-2}(x)\sec^2(x)$$

   **in preparation to use a change of variable** $u = \tan x$. We use the identity to rewrite the remaining $\sec^{n-2}(x)$ in terms of $\tan x$.

2. If $m$ is odd and $n \geq 1$, we isolate a factor of $\sec(x)\tan(x)$

   $$\tan^m(x)\sec^n(x) = \tan^{m-1}(x)\sec^{n-1}(x)\boxed{\sec(x)\tan(x)}$$

   **in preparation to use a change of variable** $u = \sec x$. We use the identity to rewrite the remaining $\tan^{m-1}(x)$ ($m-1$ is even now) in terms of $\sec x$.

3. If $m$ is even and $j$ is odd, then use the identity to express $\tan^m(x)$ in terms of $\sec(x)$. Use **Integration-by-parts** to integrate odd powers of $\sec(x)$ per Example 12.6.

4. $m$ odd and $n = 0$, i.e., pure odd power of tangent. Rewrite

   $$\tan^m(x) = \tan^{m-2}(x)\tan^2(x) = \tan^{m-2}(x)\left(\sec^2(x) - 1\right) = \tan^{m-2}(x)\sec^2(x) - \tan^{m-2}(x).$$

   Integrate the first term using a change of variable $u = \tan(x)$. Repeat the rewriting process for $\tan^{m-2}(x)$ if necessary.

## 12.3 Optional: Different Angles

From the past, we understand how to integrate

$$\int \sin(2x)\,dx$$

either via a direct inspection or a quick change of variable $u = 2x$.

$$\int \sin(2x)\cos(2x)\,dx$$

is also not so terrible because $u = \sin(2x)$ gets the job done (fill out the details yourself).

What if now a cosine counterpart shows up, with a *different* angle?

**Question 12.1.** Evaluate

$$\int \sin(2x)\cos(3x)\,dx.$$

Trust me, nothing you know from before or this lecture will work.

We take a step back and think about how these two things may be produced. Recall that there are sum and difference formulas for sine and cosine.

$$\begin{aligned}\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y)\\ \sin(x-y) &= \sin(x)\cos(y) - \cos(x)\sin(y)\end{aligned}$$

If we add these two equations, we obtain

$$\begin{aligned}\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)\\ \sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)\\ \implies \sin(\alpha+\beta) + \sin(\alpha-\beta) &= 2\sin(\alpha)\cos(\beta)\end{aligned}$$

If we let $\alpha = ax$ and $\beta = bx$, and divide both sides by 2, then we have

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a+b)\,x) + \frac{1}{2}\sin((a-b)\,x)\,.$$

This formula is quite handy because we can express a product of sine and cosine of **any** two angles, using the sum of sines.

We are now in position to solve the last example.

**Example 12.7.** Evaluate

$$\int \sin(2x)\cos(3x)\,dx.$$

**Solution.**

$$\begin{aligned}
\int \sin(2x)\cos(3x)\,dx &\overset{\text{identity}}{=} \int \frac{1}{2}\sin(2x+3x)\,dx + \int \frac{1}{2}\sin(2x-3x)\,dx \\
&\overset{\sin(-x)=-\sin(x)}{=} \frac{1}{2}\int \sin(5x)\,dx - \frac{1}{2}\int \sin(x)\,dx \\
&= \frac{1}{2}\left[-\frac{1}{5}\cos(5x)\right] - \frac{1}{2}\left[-\cos(x)\right] \\
&= -\frac{1}{10}\cos(5x) + \frac{1}{2}\cos(x) + C
\end{aligned}$$

How do we then deal with $\sin(ax)\sin(bx)$ and $\cos(ax)\cos(bx)$? The cosine sum/difference identity comes to the rescue.

$$\begin{aligned}
\cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\
\cos(\alpha-\beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)
\end{aligned}$$

Adding these two equations, we have

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta)$$

Subtracting these two equations, we have

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)$$

Using $\alpha = ax$ and $\beta = bx$, we have

$$\begin{aligned}
\cos(ax)\cos(bx) &= \frac{1}{2}\cos((a+b)\,x) + \frac{1}{2}\cos((a-b)\,x) \\
\sin(ax)\sin(bx) &= \frac{1}{2}\cos((a-b)\,x) - \frac{1}{2}\cos((a+b)\,x)
\end{aligned}$$

**Question 12.2.** Evaluate $\int \cos(2x)\cos(5x)\,dx$ and $\int \sin(6x)\cos(4x)\,dx$.

This part turns out to be extremely relevant in your future studies in Ordinary and Partial Differential Equations, whose solutions model or approximate real-world phenomena since their constructions involve first principles, such as conservation of mass, current, or energy. In short, for $0 < x < L$, where $L$ is the length of our domain, we have

$$\int_0^L \cos\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = 0, \quad m \neq n,$$

known as the *orthogonality property* of the sine and cosine functions. For any "nice" function $f(x)$, the orthogonality property enables us to find the Fourier representation of it, namely, the coefficients $a_n$ and $b_n$ that satisfy

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + c_n \sin\left(\frac{n\pi}{L}x\right).$$

Such representation helps us solve ordinary and partial differential equations. If you want to know more about this, you may take linear algebra, differential equations and then partial differential equations to fully appreciate the beauty of sines and cosines.

# 13 Lecture XIII: Trigonometric Substitutions

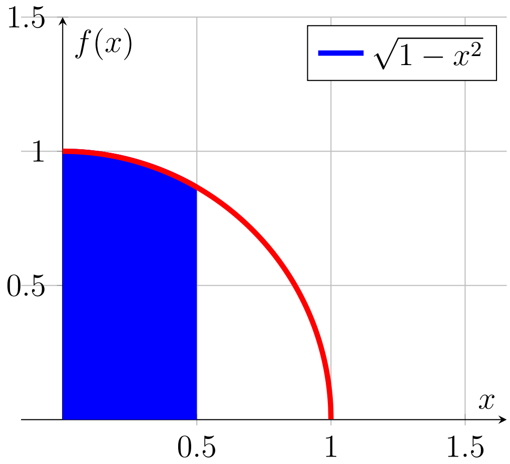
## 13.1 A Motivating Example

From a geometric standpoint, the definite integral

$$\int_0^1 \sqrt{1-x^2}dx$$

represents the area of a semicircle of radius 1. But to be able to note this geometric feature hinges on the information provided by the integrand with the prescribed bounds of integration. One becomes immediately stuck if the integral is changed to

$$\int_0^{\frac{1}{2}} \sqrt{1-x^2}dx.$$




This integral is nearly the worse kind because of the following reasons:

1. It has a single function integrand – terrible for a direct change of variable, e.g. $u = 1 - x^2$ will not work because there is nothing left for us to express the differential $du$.

2. It is not trigonometric, so nothing from the last section can help.

3. Even with integration by parts, we are forced to set $u = \sqrt{1-x^2}$ and $dv = dx$, since otherwise we are dealing with the same problem.

$$u = \sqrt{1-x^2}, \quad du = -\frac{x}{\sqrt{1-x^2}}dx;$$
$$dv = dx, \quad v = x.$$

which produces

$$\int \sqrt{1-x^2}dx = x\sqrt{1-x^2} + \int \frac{x^2}{\sqrt{1-x^2}}dx,$$

a worse situation as the ugly square root now moves to the denominator.

So, game over?

## 13.2 Method of Trigonometric Substitution

We once again embrace the famous identity

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

but express the other way around,

$$1 - \sin^2(\theta) = \cos^2(\theta).$$

How can this identity help us? Let's observe the defining similarity between the expression

$$1 - x^2 \, v.s. \, 1 - \sin^2(\theta).$$

If we set $x = \sin(\theta)$, then we can easily express

$$\begin{aligned} 1 - x^2 &= 1 - \sin^2(\theta) = \cos^2(\theta) \\ \implies \sqrt{1-x^2} &= \sqrt{1-\sin^2(\theta)} = \sqrt{\cos^2(\theta)} = \cos(\theta). \end{aligned}$$

In other words, we have changed the variable from $x$ to $\theta$, i.e., turned an $x$-problem into a $\theta$-problem. This suggests that our differential $dx$ should also be changed into an expression involving $d\theta$. This is not a difficult job if you understand differentials.

$$x = \sin(\theta) \implies dx = \cos(\theta)\, d\theta.$$

This change of variable is called the **Method of Trigonometric Substitution**. Unlike regular substitution, where we replace a part of the integrand with a new variable, here we replace the independent variable $x$ directly with a trigonometric function.

#### 13.2.1 Type $\sqrt{a^2-x^2}$ with $x = a\sin(\theta)$

**Example 13.1.** Find the antiderivative of $\sqrt{1-x^2}$, and evaluate the definite integral

$$\int_0^{\frac{1}{2}} \sqrt{1-x^2}dx.$$

**Solution.**

$$\begin{aligned}
\int \sqrt{1-x^2}dx &\overset{x=\sin(\theta)}{\underset{dx=\cos(\theta)d\theta}{=}} \int \sqrt{1-\sin^2(\theta)}\cos(\theta)\,d\theta && \text{(8a)}\\
&= \int \sqrt{\cos^2(\theta)}\cos(\theta)\,d\theta && \text{(8b)}\\
&\overset{\dagger}{=} \int \cos^2(\theta)\,d\theta && \text{(8c)}\\
&= \frac{1}{2}\int (1+\cos(2\theta))\,d\theta && \text{(8d)}\\
&= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C && \text{(8e)}
\end{aligned}$$

You may complain: shouldn't we express the antiderivative in terms of $x$? Well-said. We will do it in the next example. Let's not forget that our task here is to evaluate the definite integral.

Every change of variable will induce a change in the bounds of integration. We keep the following record:

$$\begin{aligned}
x = 0, &\implies \theta = 0;\\
x = \frac{1}{2}, &\implies \theta = \frac{\pi}{6}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^{\frac{1}{2}} \sqrt{1-x^2}dx &= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_{\theta=0}^{\theta=\frac{\pi}{6}}\\
&= \frac{1}{2}\left[\left(\frac{\pi}{6} + \frac{1}{2}\sin\left(\frac{\pi}{3}\right)\right) - (0+0)\right]\\
&= \frac{1}{2}\left(\frac{\pi}{6} + \frac{1}{2}\frac{\sqrt{3}}{2}\right)\\
&= \frac{1}{2}\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right).
\end{aligned}$$

**Remark 13.1** (The Most Important Idea in this Section)**.** *There is a † in the third step above per Equation (8). We took a square root of cosine square and immediately combined it with the cosine outside. This step requires further scrutiny because typically.*

$$\sqrt{\cos^2(\theta)} = |\cos(\theta)|$$

*Why can we just remove the absolute value sign? This is because* $\cos\theta > 0$*, but why?*

*Note that the original function* $\sqrt{1-x^2}$ *has a natural domain* $-1 \le x \le 1$*. With a change of variable* $x = \sin(\theta)$*, the bounds for* $x$ *put bounds on* $\theta$*. Replacing* $x$ *by* $\sin(\theta)$ *in the inequality and solving it leads to*

$$-1 \le \sin(\theta) \le 1 \implies -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

*This range of* $\theta$ *corresponds to the first and fourth quadrant. Guess what?* $\cos\theta$ *is positive in* ***both*** *of these quadrants, thus justifying*

$$|\cos(\theta)| = \cos(\theta).$$

**Takeaway 13.1.** The remark above seems a small thing, but is often the reason why people (yes, including mathematicians) screw up a change of variable because of the lack of care of basic mathematics.

So, how can we be more careful? Right before you proceed with the method of trigonometric substitution, consider

1. Note the domain of the integrand – this determines the range of values for $x$, which is important since you set $x =$ Some Trig Function right after.

2. Figure out the range of values for $\theta$.

3. Do proper mathematics in this new variable (just like the remark).

**Example 13.2** (A More General Example)**.** What about

$$\int \sqrt{a^2 - x^2}dx$$

for any positive number $a$?

**Solution.** Set $x = a\sin(\theta)$.

$$\begin{aligned}
\int \sqrt{a^2 - x^2}dx \quad & \underset{dx=a\cos(\theta)d\theta}{\overset{x=a\sin(\theta)}{=}} \quad \int \sqrt{a^2 - a^2\sin^2(\theta)}a\cos(\theta)\,d\theta \\
& = \quad \int \sqrt{a^2\left(1 - \sin^2(\theta)\right)}a\cos(\theta)\,d\theta \\
& = \quad \int \sqrt{a^2\cos^2(\theta)}a\cos(\theta)\,d\theta \\
& = \quad a^2\int \cos^2(\theta)\,d\theta \\
& = \quad \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C \\
& = \quad \frac{a^2}{2}\left[\theta + \sin\theta\cos\theta\right] + C
\end{aligned}$$

where we noted that the natural domain for the integrand is

$$\begin{aligned}
-a \le x \le a \implies & -a \le a\sin\theta \le a \\
\implies -1 \le \sin\theta \le 1 \implies & -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\
\implies \cos\theta > 0 &
\end{aligned}$$

justifying the square root of cosine squared as just cosine.

Here, we are NOT done because the antiderivative should be written in terms of $x$. We make use of the triangle formed by the change of variable $x = a\sin\theta$. From

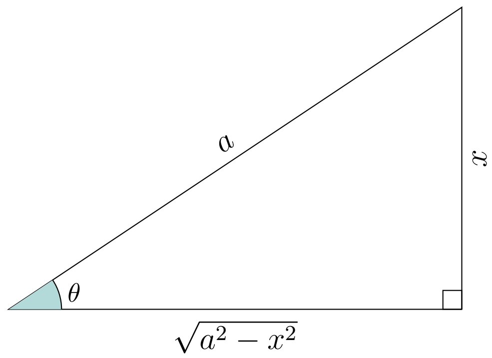



Figure 1: Schematics of the triangle induced by the change of variable $x = a\sin(\theta)$.

the antiderivative expression,

$$\int \sqrt{a^2 - x^2}dx = \frac{a^2}{2}\left[\theta + \sin\theta\cos\theta\right] + C,$$

we need to express $\theta$ and $\cos\theta$ (since we know $\sin(\theta) = \frac{x}{a}$ already via the change of variable. We find

$$\cos(\theta) = \frac{\sqrt{a^2 - x^2}}{a},$$
$$\theta = \sin^{-1}\left(\frac{x}{a}\right).$$

Thus,

$$\begin{aligned}
\int \sqrt{a^2 - x^2}dx &= \frac{a^2}{2}\left[\theta + \sin\theta\cos\theta\right] + C \\
&= \frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\frac{\sqrt{a^2 - x^2}}{a}\right] + C \\
&= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2} + C
\end{aligned}$$

**Example 13.3** (Variant)**.**

$$\begin{aligned}
\int \frac{x^2}{\sqrt{4 - x^2}}dx &\underset{dx=2\cos(x)dx}{\overset{x=2\sin(x)}{=}} \int \frac{4\sin^2(\theta)}{\sqrt{4 - 4\sin^2(\theta)}}2\cos(\theta)\,d\theta \\
&= \int \frac{4\sin^2(\theta)}{2\cos(\theta)}2\cos(\theta)\,d\theta \\
&= 4\int \sin^2(\theta)\,d\theta \\
&= 4\int \frac{1 - \cos(2\theta)}{2}d\theta \\
&= 2\left[\theta - \frac{\sin 2\theta}{2}\right] + C \\
&= 2\left[\theta - \sin(\theta)\cos(\theta)\right] + C \\
&= 2\left[\sin^{-1}(x) - \frac{x}{2}\frac{\sqrt{4 - x^2}}{2}\right] + C
\end{aligned}$$

**Strategy 13.1** (Solve integrals with $\sqrt{a^2 - x^2}$ as part of the integrand.)**.**

1. Make sure the integral cannot be solved by simpler methods.
2. Set $x = a\sin(\theta)$ and $dx = a\cos(\theta)\,dx$.
3. $\sqrt{a^2 - x^2} = a\cos\theta$.

4. Simplify (very important).
5. Evaluate the (simpler) integral now, typically a trigonometric integrals, with techniques learned from the last section.
6. Draw the triangle to re-express the antiderivative in terms of $x$.

# 14 Lecture XIV: Trigonometric Substitution cont.

Using a trigonometric substitution $x = a\sin(\theta)$, thanks to the famous identity $\sin^2(\theta) + \cos^2(\theta) = 1$, we can solve integrals involving $\sqrt{a^2 - x^2}$. What about $\int \sqrt{a^2 + x^2}dx$? The plus sign does not promote the use of this identity, but another.

## 14.1 Type $\sqrt{a^2 + x^2}$ with $x = a\tan(\theta)$

Compare the expressions

$$1 + x^2 \ v.s.\ 1 + \tan^2(\theta)$$

both of which have the form 1 plus something squared. The second item is the other famous identity

$$1 + \tan^2(\theta) = \sec^2(\theta).$$

Thus, it seems natural to use the change of variable

$$x = \tan(\theta) \implies dx = \sec^2(\theta)\, d\theta,$$

and thus

$$\begin{aligned}
\int \sqrt{1 + x^2}dx &= \int \sqrt{1 + \tan^2(\theta)} \sec^2(\theta)\, d\theta \\
&= \int |\sec\theta| \sec^2(\theta)\, d\theta \\
&= \int \sec^3(\theta)\, d\theta \\
&= \frac{\sec(\theta)\tan(\theta)}{2} + \frac{1}{2}\ln|\sec(\theta) + \tan(\theta)| + C
\end{aligned}$$

where we drop the absolute value sign because of the subtle reason (see remark in the last lecture): domain of the integrand is all real numbers, and thus

$$-\infty < x < \infty \implies -\infty < \tan\theta < \infty \implies -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

which corresponds to the 1st and 4th quadrant. $\sec\theta = \frac{1}{\cos\theta} > 0$ because $\cos\theta > 0$ in these two quadrants.

We aren't done since we need to express the answer in terms of $x$. Once again, we utilize the triangle (see Figure 2) and find that

$$\sec\theta = \sqrt{1 + x^2},$$

and therefore,

$$\begin{aligned}\int \sqrt{1+x^2}dx &= \frac{\sec(\theta)\tan(\theta)}{2} + \frac{1}{2}\ln|\sec(\theta)+\tan(\theta)| + C \\ &= \frac{x\sqrt{1+x^2}}{2} + \frac{1}{2}\ln\left|\sqrt{1+x^2}+x\right| + C.\end{aligned}$$

We likely never expected this answer from the get-go, but now we are able to say confidently that this is the answer because every step is justified.

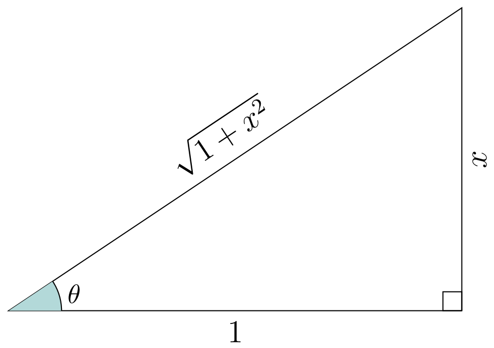



Figure 2: Schematics of the triangle induced by the change of variable $x = \tan(\theta)$.

In general, using $x = a\tan(\theta)$, we write

$$\begin{aligned}\sqrt{a^2+x^2} &= \sqrt{a^2 + a^2\tan^2(\theta)} \\ &= \sqrt{a^2\left(1+\tan^2(\theta)\right)} \\ &= \sqrt{a^2}\sqrt{\sec^2(\theta)} \\ &= a\sec(\theta).\end{aligned}$$

**Example 14.1** (Small variant)**.** Rewrite

$$\int x^3\sqrt{x^2+4}dx$$

by using a trigonometric substitution and solve it.

**Solution.** Since the expression under the square root is of the type $a^2+x^2$, we use the change of variable

$$x = 2\tan\theta \implies dx = 2\sec^2(\theta)\,d\theta.$$

Thus,

$$
\begin{aligned}
\int x^3\sqrt{x^2+4}dx &= \int 8\tan^3(\theta)\sqrt{4+4\tan^2(\theta)}d\theta \\
&= 8\int \tan^3(\theta)\,2\sec(\theta)\,d\theta \\
&= 16\int \tan^2(\theta)\sec(\theta)\tan(\theta)\,d\theta \\
&= 16\int \left(\sec^2(\theta)-1\right)\sec(\theta)\tan(\theta)\,d\theta \\
&\overset{u=\sec\theta}{=} 16\int \left(u^2-1\right)du \\
&= \frac{16}{3}u^3-16u+C \\
&= \frac{16}{3}\sec^3(\theta)-16\sec\theta+C \\
&= \frac{16}{3}\left(\frac{\sqrt{4+x^2}}{2}\right)^3-8\sqrt{4+x^2}+C.
\end{aligned}
$$

where we used the triangle with opposite $x$, adjacent 2 and thus hypotenuse $\sqrt{4+x^2}$ – $\sec\theta=\frac{\text{hypotenuse}}{\text{adjacent}}=\frac{\sqrt{4+x^2}}{2}$. See Figure 3.

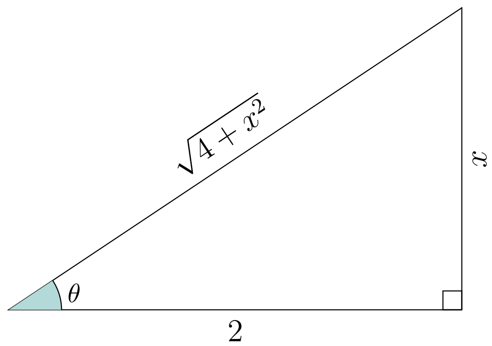



Figure 3: Schematics of the triangle induced by the change of variable $x=2\tan(\theta)$.

**Example 14.2** (Slightly harder)**.** Evaluate

$$\int \frac{x^2}{\sqrt{1+x^2}}dx.$$

**Solution.** The change of variable is $x = \tan(\theta)$, with differential $dx = \sec^2(\theta)\, d\theta$.

$$\begin{aligned}
\int \frac{x^2}{\sqrt{1+x^2}} dx &= \int \frac{\tan^2(\theta)}{\sqrt{1+\tan^2(\theta)}} \sec^2(\theta)\, d\theta \\
&= \int \tan^2(\theta) \sec(\theta)\, d\theta \\
&= \int \left(\sec^2\theta - 1\right) \sec\theta d\theta \\
&= \int \sec^3\theta d\theta - \int \sec\theta d\theta \\
&= \frac{\sec(\theta)\tan(\theta)}{2} + \frac{1}{2}\ln|\sec(\theta)+\tan(\theta)| - \ln|\sec(\theta)+\tan(\theta)| + C \\
&= \frac{x\sqrt{1+x^2}}{2} - \frac{1}{2}\ln\left|\sqrt{1+x^2}+x\right| + C.
\end{aligned}$$

where we used the known solution to the integral of secant cubed, see Example 12.6. (turns out to be more useful than it seems).

## 14.2 Optional: Type $\sqrt{x^2 - a^2}$ with $x = a\sec(\theta)$

We have successfully dealt with the case $\int \sqrt{a^2 \pm x^2} dx$. A subtle case arises:

$$\int \sqrt{x^2 - a^2} dx.$$

With such a minute flip of the terms, the technique is similar yet the change of variable is vastly different. We use the tangent/secant identity but in a rearranged form,

$$\sec^2\theta - 1 = \tan^2\theta.$$

So, with $x = a\sec\theta$, we have

$$\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\left(\sec^2\theta - 1\right)} = |a\tan\theta|. \tag{9}$$

Once again, we arrive at the subtle issue of possibly dropping the absolute value sign.

The original integrand function $\sqrt{x^2 - a^2}$ has domain $(-\infty, -a] \cup [a, \infty)$, or in other words, $x \geq a$ or $x \leq -a$. With the proposed change of variable $x = a\sec\theta$, we need to solve for the bounds of $\theta$, i.e.,

$$x \geq a \implies a\sec\theta \geq a \implies \sec\theta \geq 1 \implies \cos\theta \leq 1$$

while the other inequality suggests

$$x \leq -a \implies a\sec\theta \leq -a \implies \sec\theta \leq -1 \implies \cos\theta \geq -1.$$

Combining both statements about $\cos\theta$, namely,

$$-1 \leq \cos\theta \leq 1, \quad \text{(natural range of } \cos\theta)$$

we deduce that $0 \leq \theta \leq \pi$ (natural domain of $\cos\theta$), representing the 1st and 2nd quadrant. Guess what, we can no longer say $|\tan\theta| = \tan\theta$ for free now, because $\tan\theta$ is not always positive. In fact, it is negative in the 2nd quadrant, and positive in the 1st.

$$|\tan(\theta)| = \begin{cases} \tan(\theta), & 0 \leq \theta < \frac{\pi}{2}; \\ -\tan(\theta), & \frac{\pi}{2} < \theta \leq \pi. \end{cases}$$

From Equation (9),

$$\sqrt{x^2 - a^2} = |a\tan\theta|,$$

we are also able to tell that

$$|a\tan(\theta)| = \begin{cases} a\tan(\theta), & x \geq a \\ -a\tan(\theta), & x \leq -a \end{cases} \tag{10}$$

We will see how this can be a convenience for the example below.

**Example 14.3.** Evaluate

$$\int_1^{\sqrt{2}} \sqrt{x^2 - 1}dx.$$

**Solution.** Introduce $x = \sec\theta$ and thus $dx = \sec\theta\tan\theta d\theta$.

Notice that the bounds of integration surmise that $1 \leq x \leq \sqrt{2}$, and we identify $a = 1$ here. Therefore, by Equation (10), we have

$$\sqrt{x^2 - 1} = |\tan\theta| = \tan\theta$$

because $x \geq 1$.

With the bounds in $x$, we find the bounds in $\theta$,

$$\begin{aligned} &x = 1 \implies \sec\theta = 1 \implies \theta = 0; \\ &x = \sqrt{2} \implies \sec\theta = \sqrt{2} \implies \cos\theta = \frac{1}{\sqrt{2}} \implies \theta = \frac{\pi}{4}. \end{aligned}$$

This change of variable reduces the integrand to

$$\begin{aligned}
\int_1^{\sqrt{2}} \sqrt{x^2-1}dx &= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2\theta - 1}\sec\theta\tan\theta d\theta \\
&= \int_0^{\frac{\pi}{4}} \tan\theta\sec\theta\tan\theta d\theta \\
&= \int_0^{\frac{\pi}{4}} \tan^2(\theta)\sec\theta d\theta \\
&= \int_0^{\frac{\pi}{4}} \left(\sec^2(\theta)-1\right)\sec\theta d\theta \\
&= \int_0^{\frac{\pi}{4}} \sec^3(\theta)\, d\theta - \int_0^{\frac{\pi}{4}} \sec\theta d\theta \\
&= \left[\frac{\sec(\theta)\tan(\theta)}{2} + \frac{1}{2}\ln|\sec(\theta)+\tan(\theta)| - \ln|\sec(\theta)+\tan(\theta)|\right]_{\theta=0}^{\theta=\frac{\pi}{4}} \\
&= \left[\frac{\sec(\theta)\tan(\theta)}{2} - \frac{1}{2}\ln|\sec(\theta)+\tan(\theta)|\right]_{\theta=0}^{\theta=\frac{\pi}{4}} \\
&= \left(\frac{\sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right)}{2} - \frac{1}{2}\ln\left|\sec\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{4}\right)\right|\right) - \\
&\quad \left(\frac{\sec(0)\tan(0)}{2} - \frac{1}{2}\ln|\sec(0)+\tan(0)|\right) \\
&= \left[\left(\frac{\left(\sqrt{2}\right)(1)}{2} - \frac{1}{2}\ln\left|\sqrt{2}+1\right|\right) - \left(\frac{\cancelto{0}{\sec(0)\tan(0)}}{2} - \frac{1}{2}\cancelto{0}{\ln|1|}\right)\right] \\
&= \frac{\sqrt{2}}{2} - \frac{1}{2}\ln\left(\sqrt{2}+1\right)
\end{aligned}$$

**Takeaway.** Know your special angles. You are more likely tripped up by them than the calculus technique here.

# Trigonometric Substitution Summary

Only when you **must** use a trigonometric substitution, you consider the following change of variables according to the form

$$\begin{cases} \sqrt{a^2 - x^2}, & x = a\sin(x). \\ \sqrt{a^2 + x^2}, & x = a\tan(x). \\ \sqrt{x^2 - a^2}, & x = a\sec(x). \end{cases}$$

Less is more.

**Question 14.1.** Evaluate this integral in **three** ways.

$$\int_{-1}^{1} \frac{x}{x^2 + 1} dx.$$

**Question 14.2.** Try using $x = \cos\theta$ for

$$\int \frac{dx}{\sqrt{1 - x^2}}.$$

Compare the answer to the case where you use $x = \sin\theta$. Are they the same?

# 15 Lecture XV: Integration by Partial Fractions

This section devotes to the integrals of rational functions, $\int \frac{P(x)}{Q(x)}dx$. So far in our course, we have seen examples such as

$$\int \frac{1}{x+a}dx = \ln|x+a| + C, \tag{11}$$

or something a little harder

$$\begin{aligned}
\int \frac{1}{x^2+a^2}dx \quad &\underset{dx=a\sec^2(\theta)d\theta}{\overset{x=a\tan(\theta)}{=}} \quad \int \frac{1}{a^2\tan^2(\theta)+a^2}a\sec^2(\theta)\,d\theta \\
&= \quad \int \frac{1}{a^2(\tan^2(\theta)+1)}a\sec^2(\theta)\,d\theta \\
&= \quad \frac{1}{a}\int \frac{1}{\sec^2(\theta)}\sec^2(\theta)\,d\theta \\
&= \quad \frac{1}{a}\int d\theta \\
&= \quad \frac{1}{a}\theta + C \\
&= \quad \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.
\end{aligned}$$

## 15.1 A Motivating Example

Consider the integral

$$\int \frac{1}{1-x^2}dx.$$

A trigonometric substitution $x = \sin\theta$ will get the job done, but does it seem a bit overkill as the integrand looks so simple? There is no good reason to tell whether this problem has a simpler method, but it doesn't hurt to think about a better approach, with trigonometric substitution backing us up.

The innovation here is another mathematical reduction: we try to reduce the fraction $\frac{1}{1-x^2}$ into a form like the one in the integrand of Equation (11), i.e., a simple fraction with only a single power in the denominator. This process of reduction is called **the method of partial fraction decomposition**.

## 15.2 The Method of Partial Fraction Decomposition

This method originates from observing the process of combining simple powers using common denominators. For example

$$\text{Easier to integrate...}\underbrace{\overbrace{\frac{1}{x+2}+\frac{1}{x+3}=\frac{(x+3)+(x+2)}{(x+2)(x+3)}=\frac{2x+5}{x^2+5x+6}}^{\xrightarrow{\text{Observing putting it together}}}}_{\xleftarrow[\text{Figuring out how to break it apart}]{}}\text{...Harder to integrate}$$

Since these expressions are connected by equality, their integrals are also the same. Therefore,

$$\int \frac{2x+5}{x^2+5x+6}dx = \int \frac{1}{x+2}dx + \int \frac{1}{x+3}dx = \ln|x+2| + \ln|x+3| + C.$$

So, how do we achieve the reverse procedure, that is, given an ugly rational expression, how can we **decompose** it into **linear combinations** of simple fractions? Note that the fractions shown above both have a coefficient of 1, so in order to find the correct components of the decomposition, it amounts to determine the factors. In other words, we see coefficients $A$ and $B$ that satisfy

$$\frac{2x+5}{x^2+5x+6} = \frac{A}{x+2} + \frac{B}{x+3}.$$

Knowing how the denominator factors comes really handy. The job now is to set up two equations to solve for $A$ and $B$. We combine the fractions and find

$$\frac{A}{x+2} + \frac{B}{x+3} = \frac{A(x+3)+B(x+2)}{(x+2)(x+3)} = \frac{(A+B)x+(3A+2B)}{x^2+5x+6}$$

Since $A$ and $B$ are just real numbers, we must have

$$\begin{aligned} A+B &= 2 \\ 3A+2B &= 5 \end{aligned}$$

Multiplying the first equation by 2 and subtract it from the second, we have

$$\begin{array}{rr} & 3A+2B=5 \\ -) & 2A+2B=4 \\ \hline & A=1 \end{array}$$

and thus $B = 1$. We retrieve the **partial fraction decomposition of**

$$\frac{2x+5}{x^2+5x+6} = \frac{A}{x+2} + \frac{B}{x+3} = \frac{1}{x+2} + \frac{1}{x+3}$$

Let's return to the seemingly innocent example in the last section.

**Example 15.1.** Evaluate

$$\int \frac{1}{1-x^2} dx$$

**Solution.** The foremost task is to recognize the factors of the denominator.

$$1 - x^2 = (1-x)(1+x).$$

Now, we are in position to find the unknown coefficients in the partial fraction decomposition,

$$\boxed{\frac{1}{1-x^2}} = \frac{A}{1-x} + \frac{B}{1+x} = \frac{A(1+x) + B(1-x)}{1-x^2} = \frac{A + Ax + B - Bx}{1-x^2} = \boxed{\frac{(A-B)x + (A+B)}{1-x^2}}$$

where we employed a step of collecting common terms together, that is, terms with $x$, and constant terms.

Since the first expression (boxed) must be equal to the last expression (boxed), the numbers $A$ and $B$ must satisfy

$$\begin{array}{rr} & A - B = 0 \\ +) & A + B = 1 \\ \hline & 2A = 1 \implies A = \frac{1}{2}, \quad B = \frac{1}{2} \end{array}$$

Therefore,

$$\begin{aligned} \int \frac{1}{1-x^2} dx &= \int \frac{A}{1-x} dx + \int \frac{B}{1+x} dx \\ &= \frac{1}{2} \int \frac{1}{1-x} dx + \frac{1}{2} \int \frac{1}{1+x} dx \\ &= -\frac{1}{2} \ln|1-x| + \frac{1}{2} \ln|1+x| + C \end{aligned}$$

What if the degree in the numerator is higher than that in the denominator? Long division first (review it yourself), then decompose only when the denominator has degree more than two (inclusive).

**Example 15.2** ($\deg(P(x)) \geq \deg(Q(x))$)**.** Evaluate

$$\int \frac{x^2+3x+5}{x+1}dx$$

**Solution.** We perform long division first

$$\frac{x^2+3x+5}{x+1} = (x+2) + \frac{3}{x+1}.$$

Since the remainder has degree one in the denominator, it is already a simple fraction. We proceed with integration

$$\int \frac{x^2+3x+5}{x+1}dx = \int (x+2)\,dx + \int \frac{3}{x+1}dx = \frac{x^2}{2} + 2x + 3\ln|x+1| + C.$$

## 15.3 Nonrepeated Linear Factors

Example 15.1 belongs to the class of fractions whose denominator has **nonrepeated linear factors**, i.e., the factors all differ from each other. More precisely, we say a polynomial $Q(x)$ has nonrepeated linear factors when

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)$$

where $a_i$ and $b_i$ are respectively distinct.

Then, for a rational function $\frac{P(x)}{Q(x)}$ with $\deg(P(x)) < \deg(Q(x))$, it is possible to find constants $A_1, A_2, \ldots, A_n$ that satisfy

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}$$

**Example 15.3** (Important Example)**.** Evaluate

$$\int \frac{3x+2}{x^3-x^2-2x}dx.$$

**Solution.** Notice that the numerator has lower degree. So we begin factoring the denominator first.

$$x^3-x^2-2x = x\left(x^2-x-2\right) = x(x-2)(x+1).$$

As the factors are nonrepeating, we seek coefficients $A, B$ and $C$ satisfying

$$\frac{3x+2}{x^3-x^2-2x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1} = \frac{A(x-2)(x+1)+Bx(x+1)+Cx(x-2)}{x(x-2)(x+1)}$$

so

$$3x+2 = A(x-2)(x+1) + Bx(x+1) + Cx(x-2). \tag{12}$$

There are two ways to solve this.

1. (Method of Equating Coefficients) We did this earlier. This amounts to equating the unknown coefficients to the corresponding given coefficient with the same power of $x$. Here, we first group the terms in the numerator in Equation (12)

$$3x + 2 = A\left(x - 2\right)\left(x + 1\right) + Bx\left(x + 1\right) + Cx\left(x - 2\right)$$

$$\begin{aligned} 3x + 2 &= A\left(x^2 - x - 2\right) + B\left(x^2 + x\right) + C\left(x^2 - 2x\right) \\ &= Ax^2 - Ax - 2A + Bx^2 + Bx + Cx^2 - 2Cx \\ &= x^2\left(A + B + C\right) + x\left(-A + B - 2C\right) + \left(-2A\right) \end{aligned}$$

   So,

$$\begin{aligned} A + B + C &= 0 \\ -A + B - 2C &= 3 \\ -2A &= 2 \end{aligned}$$

   We read off from the third equation that $A = -1$, which reduces the equations to

$$\begin{aligned} B + C &= 1 \\ B - 2C &= 2 \end{aligned}$$

   This suggests that $C = -\frac{1}{3}$ and $B = \frac{4}{3}$.

2. (Method of Strategic Substitution) This is my favourite part of doing partial fractions. If we have done the decomposition correctly, the values $A, B$ and $C$ must hold for all $x$. This means we can substitute convenient values of $x$ in Equation (12) to solve for one coefficient at a time.

$$\begin{aligned} x = 0 &\implies 2 = A\left(-2\right)\left(1\right) \implies A = -1 \\ x = 2 &\implies 3\left(2\right) + 2 = B\left(2\right)\left(3\right) \implies B = \frac{4}{3} \\ x = -1 &\implies 3\left(-1\right) + 2 = C\left(-1\right)\left(-1 - 2\right) \implies C = -\frac{1}{3} \end{aligned}$$

Regardless of which method you choose, we end up with the partial fraction decomposition,

$$\begin{aligned} \int \frac{3x + 2}{x^3 - x^2 - 2x} dx &= \int \left(-\frac{1}{x} + \frac{4}{3}\frac{1}{x - 2} - \frac{1}{3}\frac{1}{x + 2}\right) dx \\ &= -\ln\left|x\right| + \frac{4}{3}\ln\left|x - 2\right| - \frac{1}{3}\ln\left|x + 2\right| + C \end{aligned}$$

# 16 Lecture XVI: Integration by Partial Fractions cont.

## 16.1 Repeated Linear Factors

If $Q(x)$ has repeated linear factors, up to *multiplicity* $n$, i.e.

$$Q(x) = (ax+b)^n,$$

then its partial fraction decomposition must contain factors up to this multiplicity,

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n} + \ldots$$

**Example 16.1.** Decompose the fraction $\frac{2x}{(x+2)^2}$.

**Solution.** Since the denominator has a repeated linear factor, we consider the decomposition

$$\frac{2x}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} = \frac{A(x+2)+B}{(x+2)^2} = \frac{Ax+(2A+B)}{(x+2)^2}.$$

Matching coefficients, we have $2x = Ax + (2A+B)$ which implies

$$\begin{aligned} A &= 2 \\ 2A + B &= 0 \implies B = -4. \end{aligned}$$

Thus,

$$\frac{2x}{(x+2)^2} = \frac{2}{x+2} - \frac{4}{(x+2)^2}$$

You may wonder – why do we need to do this in the first place, when the denominator has such a repeated factor? Why can't do we the first approach with just constant coefficient? See the optional reading at the end of this lecture.

## 16.2 Reducible and Irreducible Factors

We say a quadratic factor $ax^2 + bx + c$ is irreducible if $b^2 - 4ac < 0$, that is, the equation $ax^2 + bx + c = 0$ has no real solutions and hence cannot be decomposed into real factors. When we speak of irreducibility, we are solely referring to factors whose degree is more than 1 (e.g., quadratic, cubic, and higher). For denominator

$Q(x)$ containing irreducible quadratic factors, we must include the following term in the decomposition to account for the irreducibility,

$$\frac{Ax+B}{ax^2+bx+c}.$$

To see the reason why we do this, see the last section for an optional fun read.

**Example 16.2.** Evaluate

$$\int \frac{2x-3}{x^3+x}dx.$$

**Solution.**

$$\frac{2x-3}{x^3+x} = \frac{2x-3}{x(x^2+1)}$$

where we note that $x^2+1$ is an irreducible quadratic factor. Thus,

$$\frac{2x-3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+x(Bx+C)}{x(x^2+1)} = \frac{(A+B)x^2+Cx+A}{x(x^2+1)}$$

So, we form the equation

$$2x-3 = (A+B)x^2 + Cx + A$$

Setting $x=0$, we have $A=-3$. Since $A+B=0$, we must have $B=3$. $C=2$ since it's matching the linear coefficient. Thus,

$$\frac{2x-3}{x(x^2+1)} = -\frac{3}{x} + \frac{3x+2}{x^2+1}.$$

Altogether, we follow the integration steps

$$\begin{aligned}
\int \frac{2x-3}{x(x^2+1)}dx &= -\int \frac{3}{x}dx + \int \frac{3x+2}{x^2+1}dx \\
&= -3\ln|x| + \int \frac{3x}{x^2+1}dx + 2\int \frac{1}{x^2+1}dx \\
&\overset{u=x^2+1}{\underset{du=2xdx}{=}} -3\ln|x| + \frac{3}{2}\int \frac{1}{u}du + 2\tan^{-1}(x) + C \\
&= -3\ln|x| + \frac{3}{2}\ln|u| + 2\tan^{-1}(x) + C \\
&= -3\ln|x| + \frac{3}{2}\ln\left|x^2+1\right| + 2\tan^{-1}(x) + C
\end{aligned}$$

## 16.3 General Strategy (A Cautious Read)

**Takeaway.** For a rational function $\frac{P(x)}{Q(x)}$,

1. If $\deg\left(P\left(x\right)\right) < \deg\left(Q\left(x\right)\right)$, factor $Q\left(x\right)$.

   (a) The Method of Partial Fraction Decomposition:

      i. If $Q\left(x\right)$ has only **distinct** linear factors, then the decomposition is simple with constant coefficients.

      ii. If $Q\left(x\right)$ has a **nonrepeated** irreducible quadratic factors $ax^2+bx+c$, then its decomposition must include a linear numerator

      $$\frac{Ax+B}{ax^2+bx+c}.$$

      iii. If $Q\left(x\right)$ has a **repeated** linear factor $ax+b$ with multiplicity $n$, then the decomposition must include

      $$\frac{A_1x+B_1}{ax+b}+\frac{A_2x+B_2}{\left(ax+b\right)^2}+\cdots+\frac{A_nx+B_n}{\left(ax+b\right)^n}$$

      that is, additional fractions with denominator **up to** multiplicity $n$.

      iv. (ugly case) If $Q\left(x\right)$ has a **repeated irreducible quadratic** factors $ax^2+bx+c$ with multiplicity $n$, then the decomposition must include

      $$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{\left(ax^2+bx+c\right)^2}+\cdots+\frac{A_nx+B_n}{\left(ax^2+bx+c\right)^n}$$

2. If $\deg\left(P\left(x\right)\right) \geq \deg\left(Q\left(x\right)\right)$, do long division, and go back to step one.

## 16.4 Optional: why repeated factors?

Suppose we don't do the procedure for repeated factors, and just perform a usual decomposition as

$$\frac{2x}{\left(x+2\right)^2}=\frac{A}{x+2}+\frac{B}{x+2}=\frac{A\left(x+2\right)+B\left(x+2\right)}{\left(x+2\right)^2}=\frac{\left(A+B\right)x+2\left(A+B\right)}{\left(x+2\right)^2}.$$

Matching the coefficients, we have

$$\begin{aligned} A+B&=2\\ 2\left(A+B\right)&=0 \implies 2\times 2=0,\ \text{great!}\end{aligned}$$

In more general terms, suppose we have a fraction

$$\frac{cx+d}{(ax+b)^2} = \frac{A}{ax+b} + \frac{B}{ax+b} = \frac{A\,(ax+b)+B\,(ax+b)}{(ax+b)^2} = \frac{a\,(A+B)\,x+b\,(A+B)}{(ax+b)^2}$$

which yields the system

$$\begin{aligned} a\,(A+B) &= c \\ b\,(A+B) &= d \implies b\frac{c}{a} = d \implies ad-bc=0. \end{aligned}$$

In other words, this forces the original function to have $a, b, c$ and $d$ constrained to this equation, or otherwise we obtain obsurdity. If a method is so stringent that the function it works with has to be certain forms, it's a bad method.

In linear algebra, the quantity $ad-bc$ has a special meaning for $2\times 2$ matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

It is called the **determinant** of the matrix. When $ad-bc=0$, it implies that the vector $(a,b)$ is a multiple of $(c,d)$, i.e., there is some number $k$ such that $ax+b = k\,(cx+d)$. This suggests that one can simplify $\frac{cx+d}{(ax+b)^2}$ as one of the factors cancel, which then present no problem at all, i.e., the care we needed to take has nowhere to be taken.

## 16.5 Optional: why irreducible quadratic factors?

We see the necessity of using a factor

$$\frac{Ax+B}{ax^2+bx+c}$$

by assuming the contrary that we don't need it.

Consider the fraction

$$\frac{P\,(x)}{(\alpha x+\beta)\,(ax^2+bx+c)}, \quad a\neq 0.$$

with $(\alpha x+\beta)$ not a factor of $P\,(x)$, or otherwise they cancel, and this problem is absurd with just the quadratic factor.

Suppose that this fraction is ready to be decomposed, that is, $P\,(x)$ has a strictly smaller degree than the denominator. We may suppose that $\deg\left(P\,(x)\right)=1$, since in practice, if you claim that doing constant coefficient (as follows) is enough, then it must be enough for all possible cases. So, to logically "break" the claim, we have this degree at our disposal.

We formulate the decomposition without the linear factor, and write

$$\begin{aligned}\frac{P(x)}{(\alpha x+\beta)(ax^2+bx+c)} &= \frac{A}{\alpha x+\beta}+\frac{B}{ax^2+bx+c}\\ &= \frac{A(ax^2+bx+c)+B(\alpha x+\beta)}{(\alpha x+\beta)(ax^2+bx+c)}\\ &= \frac{Aax^2+x(Ab+B\alpha)+(Ac+B\beta)}{(\alpha x+\beta)(ax^2+bx+c)}\end{aligned}$$

OK, if $\deg(P(x)) = 1$, then the numerator is just a linear function, which means $A = 0$ (since $a \neq 0$ by assumption). But if $A = 0$, this implies that

$$\frac{P(x)}{(\alpha x+\beta)(ax^2+bx+c)} = \frac{B}{ax^2+bx+c}$$

which means $(\alpha x + \beta)$ is a factor of $P(x)$, which is not true by assumption. Thus, this decomposition cannot work.

There are simpler/prettier reasons why we must include a linear numerator in the irreducible quadratic factors when doing partial fraction decompositions. But the one given above is a practical one. You may find some explanations with a far more mathematical nature, such as the Euclidean algorithm for polynomials, but that's for another time.

# 17 Lecture XVII: Numerical Integration

So far, we have learned four main integration techniques: integration by substitution (change of variable), integration by parts, method of trigonometric substitution and lastly method of partial fraction decomposition. This class of techniques is typically known as *analytical technique*, in the sense that they only require pen-paper derivations rather than computer power.

Have we solved all possible integrals? There is always a function whose antiderivative is analytically unavailable. Rather than evaluate integrals using analytical means, we resort to various techniques of **numerical integration** to approximate their values.

## 17.1 Setup

We remind of ourselves what constitutes a Riemann sum. First, we need a partition, see definition 4.1. We state it again here.

**Definition 17.1** (Partition). *A set of points $P = x_i$ for $i = 0, 1, 2, \ldots, n$ with $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, which divides the closed interval $[a, b]$ into non-overlapping subintervals of the form, $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$, is called a* ***partition*** *of $[a, b]$. If the subintervals all have the same length, the set of points forms a* ***regular partition*** *of the interval $[a, b]$.*

We have the freedom to choose any point that lies in a subinterval as a query point $x_i^*$ such that $x_i^* \in [x_{i-1}, x_i]$; it is the point at which we evaluate the function $f(x_i^*)$. Doing so for each subinterval formulates the set of query points

$$Q = \{x_1^*, x_2^*, \ldots, x_n^*\}$$

where

$$x_{i-1} \le x_i^* \le x_i, \quad i = 1, 2, \ldots, n.$$

Recall that for the left endpoint approximation, we have $x_i^* = x_{i-1}$. For the right endpoint, we have $x_i^* = x_i$.

If the partition is regular with common subinterval length $\Delta x = \frac{b-a}{n}$, then we write each point in the partition as

$$x_i = a + i\Delta x = a + i\frac{b-a}{n}. \tag{13}$$

For the following approximation rules, all it hinges on is **where** we put the query point $x_i^*$.

## 17.2 Midpoint Approximation

By the name's sake, we put the query point in the middle of a subinterval.

$$x_i^* = \frac{x_{i-1} + x_i}{2}.$$

Noting from the expression for $x_i$ per Equation (13), we have

$$x_i^* = \frac{x_{i-1} + x_i}{2} = \frac{1}{2}\left(a + (i-1)\Delta x + a + i\Delta x\right) = \frac{1}{2}\left(2a + (2i-1)\Delta x\right) = a + \frac{2i-1}{2}\Delta x. \tag{14}$$

Then, with this formalism, we can prove that the midpoint approximation approximates the area under $f(x)$ on $[a, b]$.

**Theorem 17.1** (Midpoint Approximation)**.** *Assume that $f(x)$ is continuous on $[a, b]$. Let $n$ be a positive integer and stepsize $\Delta x = \frac{b-a}{n}$. Given a regular partition on $[a, b]$ with stepsiez $\Delta x$, and $m_i$ is the midpoint of the $i$th subinterval, set*

$$M_n = \sum_{i=1}^{n} f(m_i)\Delta x.$$

*Then*

$$\lim_{n\to\infty} M_n = \int_a^b f(x)\,dx.$$

*Proof.* Use Equation (14) and try it at home. □

**Definition 17.2** (Relative Error)**.** *Given a true answer $A$ and an approximation $B$, we say the relative error of this approximation is*

$$E_{\text{rel}} = \frac{|A - B|}{|A|}.$$

**Definition 17.3** (Query Points)**.** *Query points of an approximation are the points in an interval at which we evaluate the function.*

**Example 17.1.** Use $M_4$ to estimate $\int_0^1 (x^2+1)\,dx$.

**Solution.** Each subinterval has length $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. We list the points making the subintervals

$$\left\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}\right\}$$

The query points for the midpoint approximation then are

$$\{m_i\} = \left\{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right\}.$$

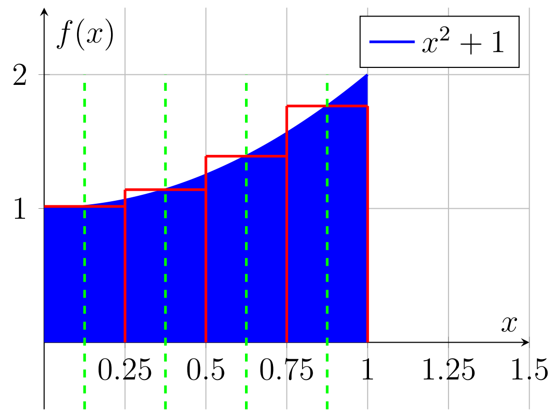
Thus, evaluating at these query points, we have

$$\begin{aligned}
M_4 &= \sum_{i=1}^{4} f\left(m_i\right) \Delta x \\
&= \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right] \\
&= \frac{1}{4}\left[\left(\frac{1}{64}+1\right) + \left(\frac{9}{64}+1\right) + \left(\frac{25}{64}+1\right) + \left(\frac{49}{64}+1\right)\right] \\
&= \frac{1}{4}\left[\frac{84}{64}+4\right] \\
&= \frac{21}{64}+1 \\
&= \frac{85}{64}.
\end{aligned}$$

We can perform an error analysis since we know the true solution

$$\int_0^1 \left(x^2+1\right) dx = \frac{4}{3} \implies E_{\text{rel}} = \frac{\left|\frac{4}{3} - \frac{85}{64}\right|}{\left|\frac{4}{3}\right|} \approx 0.00390625.$$

The midpoint approximation with just 4 subinterval is already doing pretty good. See the following sketch: red boxes use the midpoint (green dashed line) of each subinterval as height (function value), each with a width of $\frac{1}{4}$.



## 17.3 Trapezoidal Approximation

With left-endpoint, right-endpoint or midpoint, we have used *rectangles* to approximate the area under the curve. The trapezoidal approximation uses trapezoids instead, by connecting secant lines through the points in a partition.

Instead of going through the formalism, let's rework the previous example, using trapezoids to approximate the area underneath the curve.

**Example 17.2** (Trapezoidal Approximation)**.** Use $T_4$, that is, the trapezoidal approximation to estimate $\int_0^1 (x^2+1)\,dx$.

**Solution.** Denote $f(x) = x^2 = 1$. The query points of the trapezoidal rule are precisely the points in the partition. In this case, they are

$$t_i = \left\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\right\}.$$

The function values corresponding to these query points are

$$f(t_i) = \left\{1, \frac{17}{16}, \frac{5}{4}, \frac{25}{16}, 2\right\}, i = 0, 1, \ldots, 4.$$

The first trapezoid is formed by connecting the points $(0,1)$ and $\left(\frac{1}{4}, \frac{17}{16}\right)$. It has area

$$A_1 = \frac{1}{2}\left(f(t_0) + f(t_1)\right)\Delta x = \frac{1}{2}\left(1 + \frac{17}{16}\right)\frac{1}{4} = \frac{33}{128}$$

Doing a similar calculation, we find that

$$\begin{aligned}
A_2 &= \frac{1}{2}\left(f(t_1) + f(t_2)\right)\Delta x = \frac{1}{2}\left(\frac{17}{16} + \frac{5}{4}\right)\frac{1}{4} = \frac{37}{128} \\
A_3 &= \frac{1}{2}\left(f(t_2) + f(t_3)\right)\Delta x = \frac{1}{2}\left(\frac{5}{4} + \frac{25}{16}\right)\frac{1}{4} = \frac{45}{128} \\
A_4 &= \frac{1}{2}\left(f(t_3) + f(t_4)\right)\Delta x = \frac{1}{2}\left(\frac{25}{16} + 2\right)\frac{1}{4} = \frac{57}{128}
\end{aligned}$$

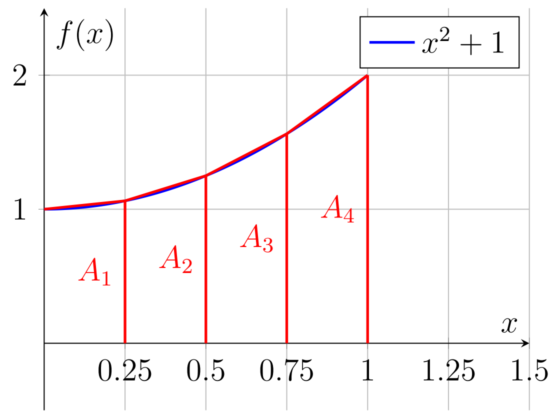
Adding up these areas, we have

$$T_4 = \frac{33}{128} + \frac{37}{128} + \frac{45}{128} + \frac{57}{128} = \frac{43}{32}$$

The relative error to the true solution $\frac{4}{3}$ is

$$E_{\text{rel}} = \frac{\left|\frac{4}{3} - \frac{43}{32}\right|}{\left|\frac{4}{3}\right|} \approx 0.0078125.$$

which is twice as bad as the relative error when using the midpoint approximation.




The calculations we just went through is awfully brute-force. However, we note a pattern in terms of the individual areas $A_i$.

$$A_i = \frac{1}{2}\left(f\left(x_{i-1}\right)+f\left(x_i\right)\right)\Delta x, \quad i = 1, 2, \ldots, n.$$

Therefore,

$$T_n = \sum_{i=1}^{n} A_i = \frac{\Delta x}{2}\sum_{i=1}^{n}\left(f\left(t_{i-1}\right)+f\left(t_i\right)\right)$$

Spelling out the sum, we note that the first and the last query point are used once, but all interior query points are evaluated twice. In other words,

$$T_n = \frac{\Delta x}{2}\left(f\left(x_0\right)+2f\left(x_1\right)+\cdots+2f\left(x_{n-1}\right)+f\left(x_n\right)\right).$$

We often associate the trapezoidal rule with the sequence of weights $\{1, 2, 2, \ldots, 2, 2, 1\}$ to help us remember how it is formed.

## 17.4 Error bounds for Midpoint and Trapezoidal Approximations

We next state a result without proof and demonstrate the way to use it. The proof relies on a topic to be learned in Calculus III (Taylor series).

**Theorem 17.2** (Error Bounds for Midpoint and Trapezoidal Approximation)**.** *Let $f(x)$ be a continuous function over $[a, b]$, having a second derivative $f''(x)$ over this interval. If $M$ is the maximum value of $f''(x)$ over $[a, b]$, then the upper bounds* *for the (absolute) error in using $M_n$ and $T_n$ to estimate $\int_a^b f(x)\,dx$ are*

$$\text{Error in } M_n \leq \frac{M(b-a)^3}{24n^3}$$
$$\text{Error in } T_n \leq \frac{M(b-a)^3}{12n^3}$$

**Example 17.3.** Determine $n$, the number of subintervals we should use to guarantee that an estimate of $\int_0^1 e^{x^2} dx$ is accurate within 0.01 if we use the midpoint rule? How about the Trapezoidal rule?

**Solution.** By the theorem, we need to identify two things: the value of $M$, and the interval $[a, b]$. The interval is easy, with $a = 0$ and $b = 1$.

The value of $M$ involves the maximum of the second derivative of the integrand function, $f(x) = e^{x^2}$. Taking a second derivative, we found that

$$f''(x) = 2e^{x^2}\left(1 + 2x^2\right).$$

On the interval $[0, 1]$, we must have

$$f''(x) = 2e^{x^2}\left(1 + 2x^2\right) \leq 2 \cdot e^{1^2} \cdot \left(1 + 2(1)^2\right) = 6e.$$

The error bound for midpoint approximation thus can be written as

$$\text{Error in } M_n \leq \frac{M(b-a)^3}{24n^3} \leq \frac{6e(1-0)^3}{24n^2} = \frac{6e}{24n^2} = \frac{e}{4n^2}.$$

In order for the error to be less than 0.01, we solve the inequality

$$\frac{e}{4n^2} \leq 0.01 \implies n \geq \sqrt{\frac{100e}{4}} \approx 8.24.$$

Since $n$ needs to be an integer, we need to round up to $n = 9$ that guarantees that

$$\left|\int_0^1 e^{x^2} dx - M_n\right| < 0.01.$$

Now, for the trapezoidal approximation error, notice that

$$T_n \leq \frac{M(b-a)^3}{12n^3} \leq \frac{6e(1-0)^3}{12n^2} = \frac{6e}{12n^2} = \frac{e}{2n^2},$$

and thus

$$\frac{e}{2n^2} \leq 0.01 \implies n \geq \sqrt{\frac{100e}{2}} \approx 11.66,$$

which means we need to round up to $n = 12$ to guarantee that

$$\left|\int_0^1 e^{x^2} dx - T_n\right| < 0.01.$$

# 18 Lecture XVIII: Numerical Integration Cont.

The midpoint approximation estimates the integral using *piecewise constant* functions – indeed, the elements of the approximation are rectangles, with a flat top. The trapezoidal approximation estimates the integral using *piecewise linear* functions – the shapes are trapezoids, with a slanted top. So, a natural follow-up question would be whether we can use *piecewise parabolas* to estimate the integral.

## 18.1 Simpson's Approximation

Let's consider a very simple case. Given the definite $\int_{x_0}^{x_2} f(x)\,dx$, we approximate it using the definite integral $\int_{x_0}^{x_2} P(x)$ where $P(x) = Ax^2 + Bx + C$ is a quadratic function passing through the points $(x_0, f(x_0)), (x_1, f(x_1))$ and $(x_2, f(x_2))$. We are able to determine the coefficients $A, B$ and $C$ using these three points. More precisely, since the parabola and the function share these three points, we set up

$$\begin{aligned} f(x_0) &= Ax_0^2 + Bx_0 + C \\ f(x_1) &= Ax_1^2 + Bx_1 + C \\ f(x_2) &= Ax_2^2 + Bx_2 + C \end{aligned}$$

We are NOT going to solve for $A, B$ and $C$, as you will see that there is no need. In addition to these equations, we note the relationship between the points $x_0, x_1$ and $x_2$, that is,

$$x_2 - x_0 = 2\Delta x, \quad 2x_1 = x_0 + x_2 (\text{since } x_1 \text{ is the midpoint of } x_0 \text{ and } x_2).$$

It is time to compute $\int_{x_0}^{x_2} P(x)\,dx$.

$$\begin{aligned}
\int_{x_0}^{x_2} f(x)\,dx &\approx \int_{x_0}^{x_2} P(x)\,dx \\
&= \int_{x_0}^{x_2} \left(Ax^2+Bx+C\right)dx \\
&= \left[\frac{A}{3}x^3+\frac{B}{2}x^2+Cx\right]_{x=x_0}^{x=x_2} \\
&= \frac{A}{3}\left(x_2^3-x_0^3\right)+\frac{B}{2}\left(x_2^2-x_0^2\right)+C\left(x_2-x_0\right) \\
&= \frac{A}{3}\left(x_2-x_0\right)\left(x_2^2+x_0x_2+x_0^2\right)+\frac{B}{2}\left(x_2-x_0\right)\left(x_2+x_0\right)+C\left(x_2-x_0\right) \\
&\overset{\text{factor }(x_2-x_0)}{=} \frac{(x_2-x_0)}{6}\left[2A\left(x_2^2+x_0x_2+x_0^2\right)+3B\left(x_2+x_0\right)+6C\right] \\
&\overset{x_2-x_0=2\Delta x}{=} \frac{\Delta x}{3}\left[2Ax_2^2+2Ax_0x_2+2Ax_0^2+3Bx_2+3Bx_0+6C\right] \\
&\overset{\mathbf{A.}}{=} \frac{\Delta x}{3}\left[\left(\underbrace{Ax_2^2+Bx_2+C}_{f(x_2)}\right)+\left(\underbrace{Ax_0^2+Bx_0+C}_{f(x_0)}\right)\right] \\
&\quad +\frac{\Delta x}{3}\left[A\left(x_2^2+2x_0x_2+x_0^2\right)+2B\left(x_2+x_0\right)+4C\right] \\
&= \frac{\Delta x}{3}\left[f(x_2)+f(x_0)+A\left(x_2+x_0\right)^2+2B\left(x_2+x_0\right)+4C\right] \\
&\overset{x_2+x_0=2x_1}{=} \frac{\Delta x}{3}\left[f(x_2)+f(x_0)+A\left(2x_1\right)^2+2B\left(2x_1\right)+4C\right] \\
&= \frac{\Delta x}{3}\left[f(x_2)+f(x_0)+4\left(\underbrace{Ax_1^2+Bx_1+C}_{f(x_1)}\right)\right] \\
&= \frac{\Delta x}{3}\left[f(x_0)+4(x_1)+f(x_2)\right].
\end{aligned}$$

where at step **A.**, we isolate terms that yield the values $f(x_0), f(x_1), f(x_2)$.

Note that this derivation holds true for **any** three consecutive points in a partition,

$$\int_{x_i}^{x_{i+2}} P(x)\,dx = \frac{\Delta x}{3}\left[f(x_i)+4f(x_{i+1})+f(x_{i+2})\right]$$

where we duff tail at $x_{i+2}$ as the point shared by the approximation on $[x_i, x_{i+2}]$ and $[x_{i+2}, x_{i+4}]$.

**Theorem 18.1** (Simpson's Approximation). *Assume that $f(x)$ is continuous over $[a,b]$. Let $n$ be a positive* ***even*** *integer and $\Delta x = \frac{b-a}{n}$. Consider a regular partition on $[a,b]$. The query points are precisely the points in the partition*

$$x_0, x_1, x_2, \ldots, x_n.$$

*Simpson's approximation with $n$ subintervals is given by*

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right].$$

*Then,*

$$\lim_{n\to\infty} S_n = \int_a^b f(x)\,dx.$$

*The approximation has error bound*

$$\textit{Error in } S_n \leq \frac{M(b-a)^5}{180n^4}$$

*where $M$ is the maximum of $\left|f^{(4)}(x)\right|$ over $[a,b]$.*

Similar to the trapezoidal rule, we remember Simpson's approximation using the sequence of weights $\{1,4,2,4,2,4,2,4,1\}$. Note that it always starts with $1,4$ and ends with $4,1$, and thus must use an **even** number of subintervals.

**Remark 18.1.** *Since Simpson's approximation uses piecewise parabolas, the approximation is* ***exact*** *for polynomials of degree up to 3 (why?). It's obvious that Simpson's rule must be exact for quadratics, but why cubics also?*

**Example 18.1.** Use $S_4$ to estimate $\int_0^1 x^3 dx$.

**Solution.** Subinterval length $\Delta x = \frac{1}{4}$. The query points are precisely the points in a partition:

$$Q = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}.$$

Simpson's approximation using 4 subintervals

$$\begin{aligned}
S_4 &= \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right) \\
&= \frac{1}{12}\left(0^3 + 4\left(\frac{1}{4}\right)^3 + 2\left(\frac{1}{2}\right)^3 + 4\left(\frac{3}{4}\right)^3 + 1^3\right) \\
&= \frac{1}{12}\left(\frac{1}{16} + \frac{1}{4} + \frac{27}{16} + 1\right) \\
&= \frac{1}{4}
\end{aligned}$$

As it turns out, this is indeed the exact answer since

$$\int_0^1 x^3 dx = \frac{x^4}{4} \mid_{x=0}^{x=1} = \frac{1}{4} - 0 = \frac{1}{4}.$$

# Part III
# Applications of Integration

For the past eighteen lectures, we have developed the theory of integration via the limit of a Riemann sum and analytical techniques such as integration by change of variable, by parts and by partial fractions. For even more nasty integrals where an analytical solution is not available, we resort to numerical integration schemes such as the midpoint, the trapezoidal or the Simpson's rule. However, all the problems we have faced are **superficial**; in other words, they are just integrals sitting there, "ready" to be solved, without *any* scientific context.

So, what's truly a scientific problem that can be addressed by integration?

## 19 Lecture XIX: Area between Curves

We have learned an awful lot about the area under a **single** curve, typically represented by the graph of a function which describes a single physical quantity, e.g., displacement. However, in many scientific problems, we are often interested in knowing the **difference** between two physical quantities, governed by two different functions. In other words, can we find the area between two curves by adapting the knowledge of the area under a single curve?

### 19.1 Scientific Motivation

Consider two particles, with velocity function $v_1(t)$ and $v_2(t)$, with $v_1 > v_2$ for all $t \geq 0$, i.e., one is going faster in the positive direction. The relative position/displacement between the two particles, up to time $t$, is captured by the integral of the velocity difference,

$$\Delta S(t) = \int_0^t [v_1(s) - v_2(s)]\, ds$$

Indeed, if $v_1$ is vastly bigger than $v_2$ in the positive direction, this integral will grow in time. We will see later that if we drop the assumption about the ordering of the velocities, a small modification is needed for this formula.

## 19.2 Area between Curves: Special Case

Given two functions $f(x)$ and $g(x)$ where $f(x) \geq g(x)$ for $x \in [a, b]$, the area between $f(x)$ and $g(x)$ can be formulated via the Riemann sum

$$A \approx \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x,$$

namely, the vertical slabs with width $\Delta x$ now have a height of the function value difference, rather than a single function.

As $n \to \infty$ (and simultaneously as $\Delta x \to 0$), we have the following result.

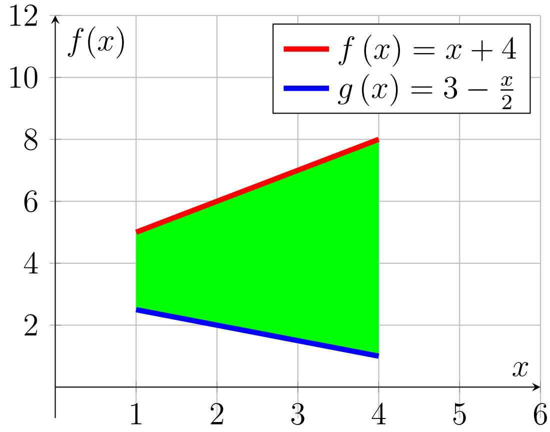
**Theorem 19.1** (Special Formula for Area between **Ordered** Curves)**.**
*Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ for $x \in [a, b]$. Let $R$ denote the region bounded above by the graph of $f(x)$ and below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of $R$ is given by*

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

**Example 19.1** (Obvious order of $f(x)$ and $g(x)$)**.**
If $R$ is the region bounded above by the graph of the function $f(x) = x + 4$ and below by the graph of the function $g(x) = 3 - \frac{x}{2}$ over the interval $[1, 4]$, find the area of $R$.

**Solution.** How can we prove that $f(x) \geq g(x)$ on the interval $[1, 4]$, as it is essential to use the Theorem? I will leave it to you. The hint is that to prove $f(x) \geq g(x)$ on $[1, 4]$ is to first show that $f(1) \geq g(1)$ and then show that $f'(x) \geq g'(x)$, namely, given that one player starts ahead (position), it also runs faster (velocity).

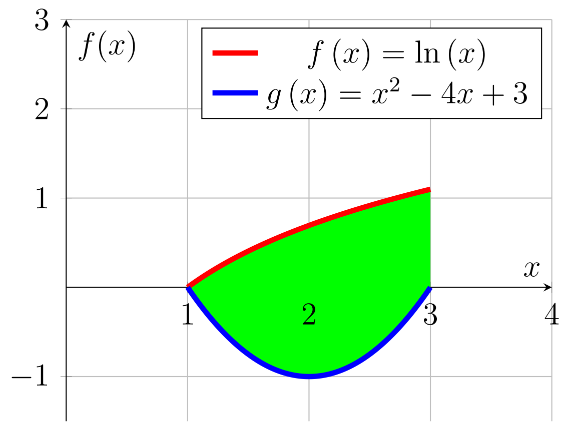
We can also just sketch the graphs (see shaded area in green). Thus, by the Theorem, the area is given by




$$\begin{aligned}
A &= \int_a^b [f(x) - g(x)]\, dx \\
&= \int_1^4 \left[(x + 4) - \left(3 - \frac{x}{2}\right)\right] dx \\
&= \int_1^4 \left[\frac{3x}{2} + 1\right] dx \\
&= \left[\frac{3}{4}x^2 + x\right]_{x=1}^{x=4} \\
&= \left(\frac{3}{4}(4)^2 + 4 - \frac{3}{4} - 1\right) \\
&= \frac{57}{4}
\end{aligned}$$

**Example 19.2** (A nonlinear example)**.**
Find the area between the curves $f(x) = \ln(x)$ and $g(x) = x^2 - 4x + 3$ on $[1, 3]$.

**Solution.** We need to first identify which function is the larger one. Since $g(x)$ is a parabola that opens upward with roots at $x = 1$ and $x = 3$, the parabola is below the $x$-axis on $[1, 3]$. Meanwhile, $f(x)$ is positive for $x \geq 1$. So in this case, $f(x) \geq g(x)$ on $[1, e]$.



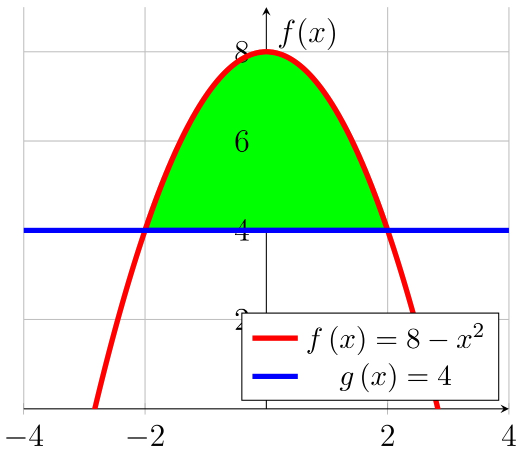

$$
\begin{aligned}
A &= \int_a^b [f(x) - g(x)]\, dx \\
&= \int_1^3 \left[\ln x - \left(x^2 - 4x + 3\right)\right] dx \\
&= \left[x \ln x - x - \frac{x^3}{3} + 2x^2 - 3x\right]_{x=1}^{x=3} \\
&= \left[3 \ln 3 - 3 - \frac{3^3}{3} + 2(3)^2 - 3(3)\right] \\
&\quad - \left[1 \cancelto{0}{\ln 1} - 1 - \frac{1}{3} + 2 - 3\right] \\
&= 3 \ln 3 - \frac{2}{3}
\end{aligned}
$$

**Example 19.3** (What if the interval is not provided?)**.**
Find the area between the curves $f(x) = 8 - x^2$ and $g(x) = 4$.

**Solution.** The foremost task is to figure out the bounds of integration. This amounts to finding the **intersection** of the two functions by setting the functions equal.

$$
\begin{aligned}
f(x) &= g(x) \\
\implies 8 - x^2 &= 4 \\
\implies x^2 &= 4 \\
\implies x &= \pm 2
\end{aligned}
$$

For $-2 \leq x \leq 2$, we note that the upper function is $f(x)$. Thus,




$$
\begin{aligned}
A &= \int_a^b [f(x) - g(x)]\, dx \\
&= \int_{-2}^{2} \left[\left(8 - x^2\right) - 4\right] dx \\
&= \int_{-2}^{2} \left(4 - x^2\right) dx \\
&= 2 \int_0^2 \left(4 - x^2\right) dx \\
&= 2 \left[4x - \frac{x^3}{3}\right]_{x=0}^{x=2} \\
&= 2 \left[8 - \frac{2^3}{3} - 0 + 0\right] \\
&= \frac{32}{3}
\end{aligned}
$$

**Question 19.1.** Read Example 2.2 in the textbook.

**Takeaway 19.1** (Solution Strategy for the Special Case)**.**

1. Identify the bounds of integration.
   (a) If given, go to step two.
   (b) If not given, then set the functions equal and find the bounds.
2. Identify which function is upper and lower.
3. Apply the formula in the Theorem.

## 19.3 Area between Curves: General Case

The condition on $f(x) \geq g(x)$ is rather restrictive because in practice, functions cross each other so the larger function becomes the smaller one. What should we do in this situation? Let's discover it with a simple case.

Suppose $f(x) \geq g(x)$ on $[a,b]$ and $g(x) \geq f(x)$ on $[b,c]$. Note that the area between $f(x)$ and $g(x)$ satisfy

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]\,dx + \int_b^c [g(x) - f(x)]\,dx \\ &= \int_a^b [f(x) - g(x)]\,dx + \int_b^c -[f(x) - g(x)]\,dx \\ &= \int_a^b |f(x) - g(x)|\,dx + \int_b^c |f(x) - g(x)|\,dx \\ &= \int_a^c |f(x) - g(x)|\,dx \end{aligned}$$

where we used the fact that the absolute value take on the negative sign if the argument is negative. This observation, in fact, provides a more general characterization of area between curves.

**Theorem 19.2** (General Formula for Area between Curves)**.**
*Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a,b]$. Let $R$ denote the region between the graphs of $f(x)$ and $g(x)$, and bounded by the lines $x = a$ and $x = b$, respectively. Then, the area of $R$ is given by*

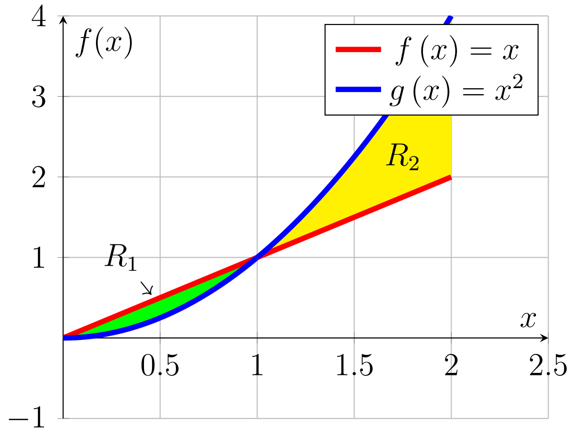
$$A = \int_a^b |f(x) - g(x)|\,dx.$$

**Example 19.4** (A Rudimentary One)**.** Find the area between the functions $f(x) = x$ and $g(x) = x^2$ on $[0,2]$.

**Solution.** By noting the fact that $x \geq x^2$ for $0 \leq x \leq 1$ and $x \leq x^2$ for $1 \leq x \leq 2$, we know there is more to this problem than it appears. Indeed, we appeal to the general case formula because the functions cross each other once, precisely, at the intersection point, found by setting the functions equal

$$\begin{aligned} f(x) &= g(x) \\ \implies x &= x^2 \\ \implies x - x^2 &= 0 \\ \implies x(1-x) &= 0 \\ \implies x &= 0, 1 \end{aligned}$$

So, the "watershed" point is $x = 1$, as it is on the interior of our domain of interest (that is, $[0, 2]$). We analyze the two subregions, $R_1$ and $R_2$, separately,




$$
\begin{aligned}
A &= \int_0^2 \left|x - x^2\right| dx \\
&= \int_0^1 \left|x - x^2\right| dx + \int_1^2 \left|x - x^2\right| dx \\
&= \underbrace{\int_0^1 \left(x - x^2\right) dx}_{R_1} + \underbrace{\int_1^2 \left(x^2 - x\right) dx}_{R_2} \\
&= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{x=0}^{x=1} + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{x=1}^{x=2} \\
&= \left[\frac{1}{6} - 0\right] + \left[\left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right] \\
&= 1
\end{aligned}
$$

**Question 19.2** (A More Involved Example)**.** Read Example 2.3 in the textbook.

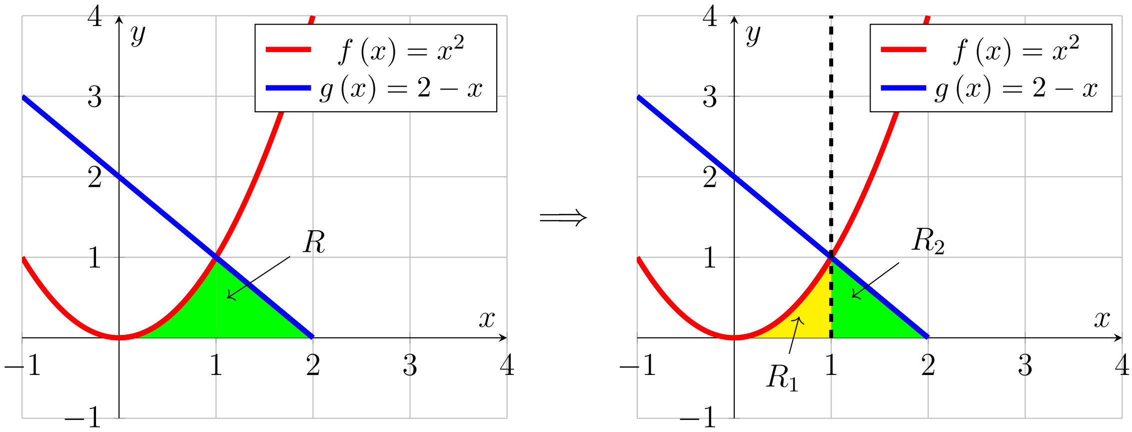
## 19.4 The Needle Method for More Complex Regions

The needle method is a term coined by Dr. Binan Gu during his lecture for Calculus IV students at WPI. The reason why it shows up now in Calculus II is because the method works quite similarly in a simpler context (for now). Let's begin with an example, understood in three ways: 1) geometric observation; 2) an application of the needle method in the $x$-direction and 3) in the $y$-direction.

**Example 19.5** (A Complex Region)**.**
Consider the region depicted in the following figure. Find the area of $R$, adjacent to the function $f(x) = x^2$, $g(x) = 2 - x$ and the horizontal line $y = 0$.

**Solution.** By a purely geometric observation with the dashed black line, we note that the area can be decomposed into two parts: 1) the area strictly under the parabola

$f(x)$, and 2) the area strictly under the line $g(x) = 2 - x$.




Therefore, total area is the sum of the individual pieces.

$$\begin{aligned} A_R &= A_{R_1} + A_{R_2} \\ &= \int_0^1 x^2 dx + \int_1^2 (2 - x)\, dx \\ &= \frac{1}{3} + \left[2x - \frac{x^2}{2}\right]_{x=1}^{x=2} \\ &= \frac{5}{6} \end{aligned}$$

**Remark 19.1** (An Important Remark leading to the Needle Method)**.** *What's missing here, without harm, is the "lower function". But didn't we just give a formula of the area of between curves, yet in the last example, we didn't quite see a difference?*

*The lower function, in the previous example, is precisely the horizontal $y = 0$. This means, the integrals in the second line of the calculations should be more "pedantically" written as*

$$A_{R_1} + A_{R_2} = \int_0^1 \left(x^2 - 0\right) dx + \int_1^2 \left[(2 - x) - 0\right] dx.$$

**Question 19.3.** But this raises a question: can we deal with integrals bounded by nonlinear curves in a more systematic fashion? How are we going to determine the bounds of integration, and which function is the upper and lower? Does the upper and lower relationship change as we jump over an intersection point? See the following example, slightly modified from the previous one. You will see that the previous "pedantic" observation is helpful and should be the go-to methodology.

**Theorem 19.3** (The Needle Method: $x$-direction.)**.**
*Let $R$ be the region between $f(x)$ and $g(x)$, with multiple intersections $x_1 < x_2 < \ldots < x_n$. We insert a vertical line, "the $x$-needle", $x = c$ through the region where $x_i \leq c \leq x_{i+1}$. Then,*

1. *The first curve this needle touches (and enters the region) is the lower function.*

2. *The second curve this needle touches (and exits the region) is the upper function.*

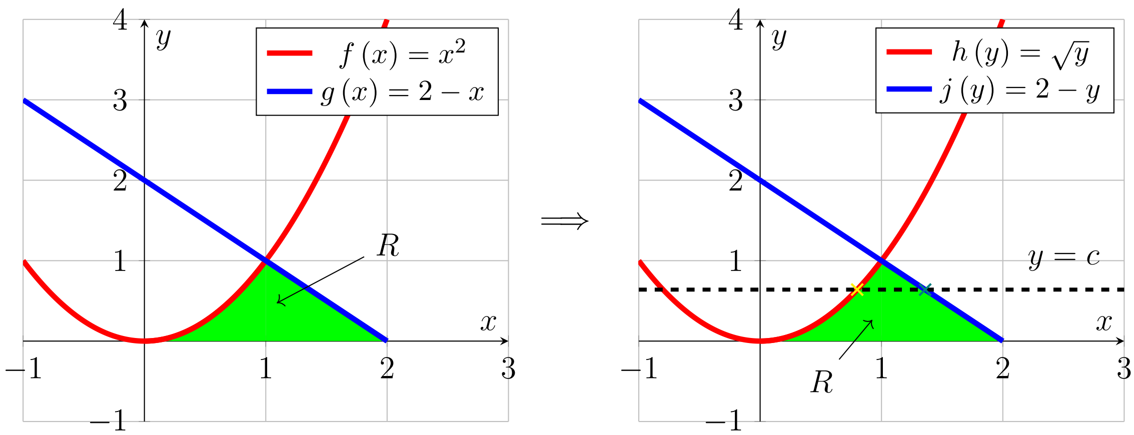
**Theorem 19.4** (The Needle Method: $y$-direction.)**.**
*Let $R$ be the region between $f(y)$ and $g(y)$, with multiple intersections $y_1 < y_2 < \ldots < y_n$. We insert a horizontal line, "the $y$-needle", $y = c$ through the region where $y_i \leq c \leq y_{i+1}$. Then,*

1. *The first curve this needle touches (and enters the region) is the lower function.*

2. *The second curve this needle touches (and exits the region) is the upper function.*

**Example 19.6** (A Complex Region Revisited)**.**
We consider the same region as in the last example. Instead of solving the problem in the $x$-variable, where a decomposition is needed, we solve the problem in the $y$-variable.

**Solution.**




The $y$-needle enters the region at the yellow cross, at the "lower" curve, now expressed in terms of $y$, that is, $h(y) = \sqrt{y}$ (instead of $y = x^2$). The $y$-needle exits the region at the teal cross, at the "upper" curve, also expressed in terms $y$, $j(y) = x = y - 2$ (instead of $y = x - 2$).

The bounds of integration, for $y$, is determined by the max and min of possible $y$-values, which is $0 \leq y \leq 1$, as you can observe from the plot.

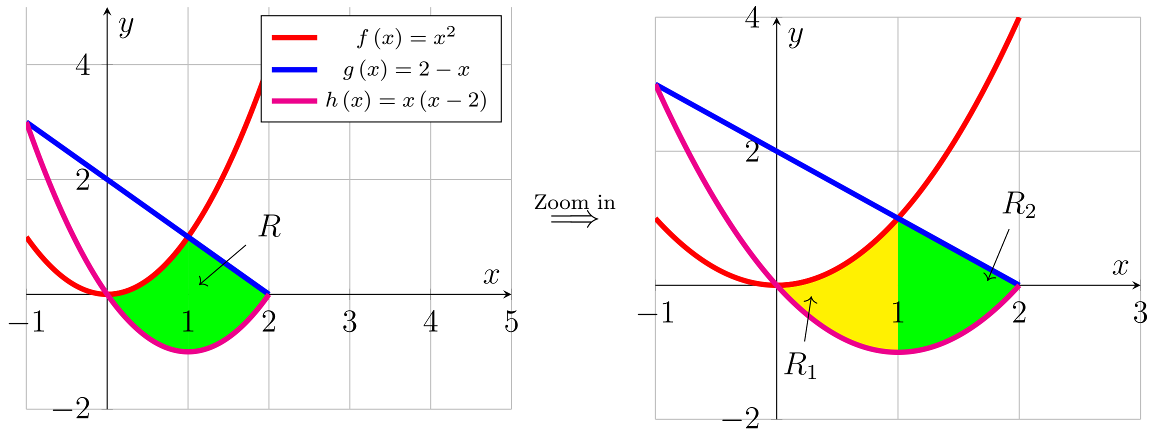
Therefore, total area is appealing to Theorem 19.1 but in the $y$-direction.

$$\begin{aligned} A &= \int_c^d \left[j\left(y\right) - h\left(y\right)\right] dy \\ &= \int_0^1 \left[2 - y - \sqrt{y}\right] dy \\ &= \left[2y - \frac{y^2}{2} - \frac{2}{3} y^{\frac{3}{2}}\right]_{y=0}^{y=1} \\ &= \left[2 - \frac{1}{2} - \frac{2}{3}\right] \\ &= \frac{5}{6} \end{aligned}$$

This is precisely the same answer as we obtained in the previous example, as expected.

## 19.5 Read after class: More Involved Examples

**Example 19.7** (A Much More Complex Region)**.**
Find the area of $R$, adjacent to the function $f\left(x\right) = x^2$, $g\left(x\right) = 2 - x$ and $h\left(x\right) = x\left(x-2\right)$, in the first and fourth quadrant, as depicted below.




**Solution.** The middle intersection point is just between $f\left(x\right)$ and $g\left(x\right)$, so it is still $x = 1$. The left intersection is between $f\left(x\right)$ and $h\left(x\right)$ at $x = 0$, while the right intersection is between $g\left(x\right)$ and $h\left(x\right)$ at $x = 2$. Thus, the region of interest can be decomposed into two parts.

$$\begin{aligned}
A &= A_{R_1} + A_{R_2} \\
&= \int_0^1 \left[f\left(x\right) - h\left(x\right)\right] dx + \int_1^2 \left[g\left(x\right) - h\left(x\right)\right] dx \\
&= \int_0^1 \left[x^2 - x\left(x-2\right)\right] dx + \int_1^2 \left[2 - x - x\left(x-2\right)\right] dx \\
&= \int_0^1 \left[x^2 - x^2 + 2x\right] dx + \int_1^2 \left[2 - x - x^2 + 2x\right] dx \\
&= \int_0^1 2x dx + \int_1^2 \left[2 + x - x^2\right] dx \\
&= \left[x^2\right]_{x=0}^{x=1} + \left[2x + x^2 - \frac{x^3}{3}\right]_{x=1}^{x=2} \\
&= 1 + \left[\left(4 + 4 - \frac{8}{3}\right) - \left(2 + 1 - \frac{1}{3}\right)\right] \\
&= \frac{11}{3}
\end{aligned}$$

# 20 Lecture XX: Determining Volumes by Slicing

The volume of a circular cylinder is rather easy to compute and to visualize, solely because of its constant cross-section. Indeed,

$$V_{\text{cylinder}} = \pi R^2 h$$

where $R$ and $h$ are the radius and the height of the cylinder, respectively.

But the volume for the cone is a dubious one,

$$V_{\text{cone}} = \frac{1}{3}\pi R^2 h$$

where $R$ is the radius of the base. It looks like quite similar to the formula of the cylinder but why the factor 1/3 and not anything else? Via calculus, we can fully justify the formula for the cone, among many, using the method of slicing.

## 20.1 The Method of Slicing: Formula for the Cone

Let's "situate" the cone in the $xy$-plane, by putting its base adjacent to the $y$-axis. The center of the base circle sits at the origin, and extend $h$ units to the tip on the $x$-axis. Viewed in the $xy$-plane, the cone has a triangular size view.

Consider an arbitrary circular cross-section of the cone, taken at $x = s$. We want to calculate the cross-section area at this very slice. Via the similar triangle, we find that the radius $r$ (lowercase) at this slice satisfies

$$\frac{r}{R} = \frac{h-s}{h} \implies r = \frac{(h-s)\,R}{h}.$$

Therefore, the area of this slice at $x = s$ satisfies

$$A\,(s) = \pi r^2 = \pi \left(\frac{(h-s)\,R}{h}\right)^2.$$

Now, if we give this sliced "disk" a thickness of $ds$, then we can compute its infinitesimal volume

$$dV\,(s) = A\,(s)\,ds.$$

Altogether, the volume of the cone is adding up all such disks, via

$$\begin{aligned}
V &= \int_0^h dV(s) \\
&= \int_0^h A(s)\,ds \\
&= \int_0^h \pi\left(\frac{(h-s)R}{h}\right)^2 ds \\
&= \frac{\pi R^2}{h^2}\int_0^h \left(h^2 - 2hs + s^2\right) ds \\
&= \frac{\pi R^2}{h^2}\left[h^2 s - hs^2 + \frac{s^3}{3}\right]_{s=0}^{s=h} \\
&= \frac{\pi R^2}{h^2}\left[h^3 - h^3 + \frac{h^3}{3}\right] \\
&= \frac{1}{3}\pi R^2 h
\end{aligned}$$

Note that this is precisely saying the volume of the circular cone is $V = \frac{1}{3}\times$base area$\times$ height.

Based on the insight of $V = \int dV = \int A(x)\,dx$, we can now compute the volume of many shapes that have an axial symmetry like the cone and the cylinder. All it amounts to is to compute the area $A(x)$ at an arbitrary $x$-value, using given information.

**Example 20.1** (Cone with a Square Base: Example 2.6 in the text)**.** Consider now a cone with a square base of side length $l$ and a height of $h$. We can find its volume via the method of slicing as well.

**Solution.** We situate the cone with a square base with its tip at the origin, and have $x$-axis act as the axis of symmetry (see sketch Figure 2.14 in the text). At an arbitrary $x$-value, we need to determine the cross-section area, which is a square of unknown side length $s$.

Via similar triangle, again, we find that this unknown side length satisfies

$$\frac{s}{l} = \frac{x}{h} \implies s = \frac{xl}{h} \implies A(x) = \left(\frac{xl}{h}\right)^2.$$

Therefore, by the method of slicing, we have

$$\begin{aligned}
V &= \int_0^h dV\,(x) \\
&= \int_0^h \left(\frac{xl}{h}\right)^2 dx \\
&= \frac{l^2}{h^2}\int_0^h x^2 dx \\
&= \frac{l^2}{h^2}\left[\frac{x^3}{3}\right]_{x=0}^{x=h} \\
&= \frac{l^2}{h^2}\left[\frac{h^3}{3} - 0\right] \\
&= \frac{1}{3}l^2 h.
\end{aligned}$$

Note that this formula makes sense because it mimics that for the circular cone, in the sense that $V = \frac{1}{3} \times$ base area $\times$ height.

# 21 Lecture XXI: Disk and Washer; Arc Length and Surface Area

## 21.1 The Disk Method

The cone and the cynlinder are often referred to as a solid of revolution. Indeed, a cone can be formed by rotating a line through the origin about the $x$-axis (or the $y$-axis). Similar, a cylinder can be obtained by rotating a constant function (horizontal line) about the $x$-axis (or a vertical line about the $y$-axis). The function we use to rotate certainly can be more general. The method of finding the volume of a solid of revolution is a special case of the method of slicing, call the "Disk Method" because the slices are in fact disks, i.e., a circle with tiny thickness.

**Example 21.1** (The Disk Method)**.** Consider the function $f(x) = x^2$ on $[0, 2]$. Revolve the parabola about the $x$-axis once to form a solid "Sombrero". Compute the volume of this solid.

**Solution.** So, all we need is the area of a slice at an arbitrary location $x$. Since the function is rotated about the $x$-axis, the function height $f(x)$ is used as the radius. Therefore,

$$A(x) = \pi (f(x))^2 .$$

Altogether,

$$\begin{aligned} V &= \int_0^2 dV(x) \\ &= \int_0^2 \pi [f(x)]^2 \, dx \\ &= \pi \int_0^2 x^4 dx \\ &= \pi \left[ \frac{x^5}{5} \right]_{x=0}^{x=2} \\ &= \frac{32}{5}\pi \end{aligned}$$

**Theorem 21.1** (The Disk Method)**.** *Let $f(x)$ be continuous and nonnegative. Define $R$ as the region bounded above by the graph of $f(x)$, below by the $x$-axis, on the left by the line $x = a$ and on the right by $x = b$. Then, the volume of the solid of revolution formed by **revolving** $R$ **around the** $x$**-axis** is given by*

$$V = \int_a^b \pi [f(x)]^2 \, dx.$$

*If $g(y)$ is* ***revolved about the y-axis*** *for $c \le y \le d$, then the volume of the solid of revolution formed in this fashion is given by*

$$V = \int_c^d \pi [g(y)]^2 \, dy.$$

**Example 21.2** (Rotating about the $y$-axis)**.** Consider the region between the function $y = 4 - x^2$ and $y = 4$ on $[0, 2]$. Rotate this region about the $y$-axis. Compute the volume of the resulting solid of revolution.

**Solution.** Let's set up the problem in the $y$-coordinate, appealing to the second part of the theorem.

$$y = 4 - x^2 \implies g(y) = x = \sqrt{4 - y}, \quad 0 \le y \le 4.$$

The slices now are horizontal, i.e., parallel to the $x$-axis, making them much easier to visualize. At an arbitrary value $y$, the disk has radius $g(y)$ (the function "height" measured from the $y$-axis to the graph of $g$). Thus,

$$\begin{aligned} V &= \int_c^d \pi [g(y)]^2 \, dy \\ &= \pi \int_0^4 \left(\sqrt{4-y}\right)^2 dy \\ &= \pi \int_0^4 (4 - y) \, dy \\ &= \pi \left[4y - \frac{y^2}{2}\right]_{y=0}^{y=4} \\ &= \pi [16 - 8] \\ &= 8\pi \end{aligned}$$

**Question 21.1.** Read Example 2.8 and 2.9 in the text.

## 21.2 The Washer Method

So far, the solids of revolution we saw have no cavities in the middle. They are produced by regions under a curve adjacent to either principal axis. However, when such region is no longer bounded below by the $x$-axis, for example, the solids of revolution will create a hole.

**Example 21.3** (Minimal Example of a Solid with a Cavity)**.** Consider the function $f(x) = \sqrt{x}$ and $g(x) = 1$ (horizontal line). Revolve the region between $f(x)$ and $g(x)$ on $[1, 4]$.

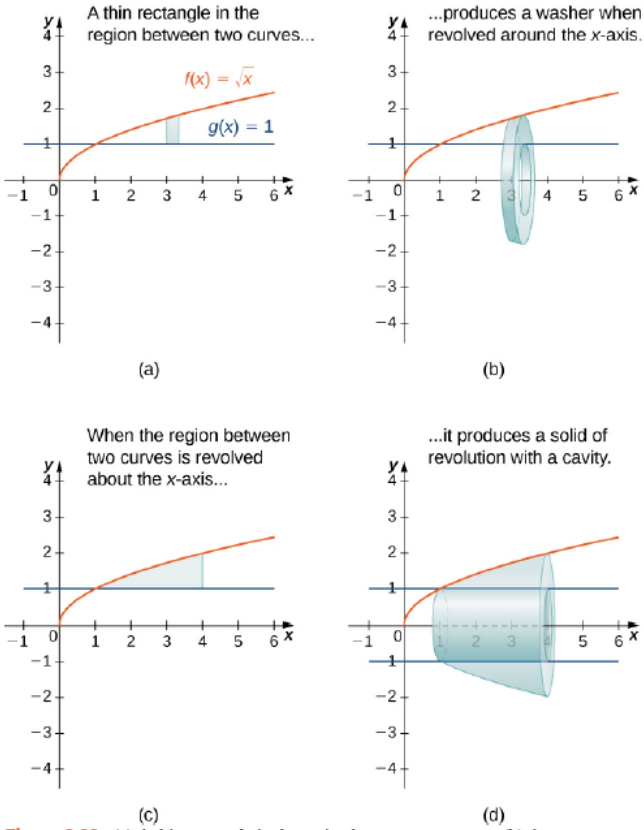



**Figure 2.22** (a) A thin rectangle in the region between two curves. (b) A representative disk formed by revolving the rectangle about the $x$-axis. (c) The region between the curves over the given interval. (d) The resulting solid of revolution.

**Solution.** We refer to the great figure in the text. We first consider a thin strip at $x$ of width $dx$ in the region of interest. If rotated about the $x$-axis, the thin strip will produce a hollow cylinder with infinitesimal volume

$$dV(x) = A(x)\,dx = \left[\pi(f(x))^2 - \pi(g(x))^2\right] dx$$

as the cross-sectional area is simply the difference of two concentric circles.

Thus,

$$\begin{aligned}
V &= \int_1^4 dV(x) \\
&= \int_1^4 \left[\pi(f(x))^2 - \pi(g(x))^2\right] dx \\
&= \pi \int_1^4 \left[\left(\sqrt{x}\right)^2 - 1^2\right] dx \\
&= \pi \int_1^4 (x-1)\,dx \\
&= \pi \left[\frac{x^2}{2} - x\right]_{x=1}^{x=4} \\
&= \pi \left[\frac{4^2}{2} - 4 - \left(\frac{1}{2} - 1\right)\right] \\
&= \frac{9}{2}\pi
\end{aligned}$$

The solid with a middle cavity resembles the washer, which we use to coin this method. In general, we collect the result in the following Theorem.

**Theorem 21.2** (The Washer Method for Solids of Revolution)**.** *Let $f(x)$ and $g(x)$ be continuous and nonnegative such that $f(x) \geq g(x)$ over $[a,b]$. Denote $R$ as the region bounded above by $f(x)$ and below by $g(x)$, on the left by $x = a$ and on the right by $x = b$. Then, the volume of the solid of revolution formed by revolving $R$ around the $x$-**axis** is given by*

$$V = \int_a^b \left[\pi(f(x))^2 - \pi(g(x))^2\right] dx$$

*For the $y$-variable counterpart: let $u(y)$ and $v(y)$ be continuous and nonnegative functions of $y$ such that $u(y) \geq v(y)$ on $[c,d]$ ($y$-values). Denote $Q$ as the region bounded above by $y = d$ and below by $y = c$, on the left by $v(y)$ and on the right by $u(y)$. Then, the volume of the solid of revolution formed by revolving $R$ around the $y$-**axis** is given by*

$$V = \int_c^d \left[\pi(u(y))^2 - \pi(v(y))^2\right] dy.$$

**Question 21.2.** Exercise for you: consider the region between $f(x) = x$, $g(x) = \frac{1}{x}$, $x = 1$ and $x = 4$. Revolve this region about the $y$-axis. Find the volume of the resulting solid of revolution.

**Example 21.4** (Axis of rotation is other than the principal axes)**.** Consider the region between the function $f(x) = 4 - x$ and the principal axes. Rotate this region about the line $y = -2$. Find the volume of the solid of revolution.

**Solution.** You may ask, can we just shift everything up by 2 units and apply the washer method? Yes. Go try it.

What I wanted to do is to directly characterize the radius at an arbitrary point $0 \leq x \leq 4$ (why 4?). The distance from the curve to the $x$-axis, at some point $x$, is precisely the function value $4 - x$. But since the axis of rotation is 2 units further out, we need to take it into account. In other words, the radius is $(4 - x) + 2 = 6 - x$. Thus, at this $x$, the cross-sectional area is

$$A(x) = \pi\left((6 - x)^2 - 2^2\right).$$

This suggests that the volume is no other than the integral of the cross-sectional area.

$$\begin{aligned} V &= \int_0^4 A(x)\,dx \\ &= \int_0^4 \pi\left[(6 - x)^2 - 2^2\right]dx \\ &= \pi \int_0^4 \left[x^2 - 12x + 32\right]dx \\ &= \pi\left[\frac{x^3}{3} - 6x^2 + 32x\right]_{x=0}^{x=4} \\ &= \frac{160\pi}{3} \end{aligned}$$

## 21.3 Arc Length

The previous section uses the Fundamental Theorem of Calculus on volume computation, namely,

$$V = \int_a^b dV(x).$$

In other words, we find the formula expressing a volume element/differential at an arbitrary location $x$, and simply "add up" (integrate) the volume differential to obtain total volume.

In a similar fashion, in order to calculate total arc length of a curve, we figure out how to express a length element/differential $ds(x)$ at an arbitrary location $x$ on the curve, and then add up all the length elements at all $x$-values.

Consider a point $(x, y)$ and its nearby point with $x$-coordinate $x + dx$ where $dx$ is arbitrarily small. The point $x + dx$ corresponds to a function value $y + dy$. Visually, we can connect the data points $(x, y)$ and $(x + dx, y + dy)$. Call the length of this secant line $ds(x)$, namely, the length element at $x$. Thus, by Pythagorus, we have

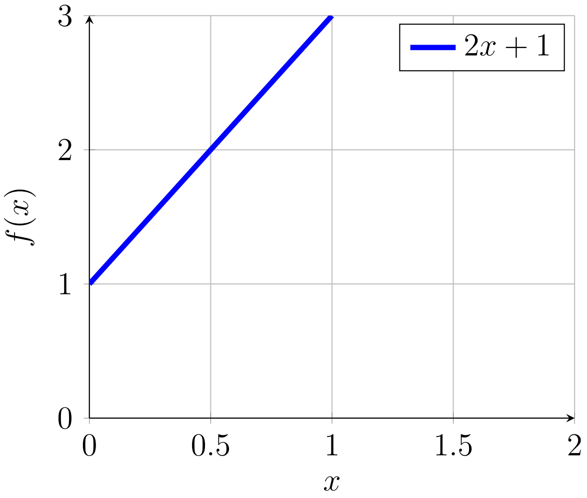
$$ds(x) = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(dx)^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right)} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Therefore, total arc length of a curve $f(x)$ on $[a, b]$ is

$$S = \int_a^b ds(x) = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + (f'(x))^2} dx.$$

The requirement is that $f(x)$ must have a well-defined first derivative.

**Example 21.5** (Sanity check for straight lines)**.** Consider the straight line $y = 2x+1$ for $m \neq 0$. Compute the arc length of this line from $[0, 1]$.




**Solution.** We know from the basic geometry that the length of this segment is $\sqrt{1^2 + 2^2} = \sqrt{5}$.

To use calculus, we find that $f'(x) = 2$. Thus,

$$L = \int_0^1 \sqrt{1 + (f'(x))^2} dx = \int_0^1 \sqrt{1 + (2)^2} dx = \sqrt{5} \int_0^1 dx = \sqrt{5}.$$

**Example 21.6** (Simple Curve - Complicated Answer)**.** Find the arc length of the function $f(x) = x^2$ on $\left[0, \frac{1}{2}\right]$.

**Solution.** Looking at the formula, we need the **derivative squared**. So

$$f'(x) = 2x \implies (f'(x))^2 = 4x^2.$$

Thus, total arc length on this interval is given by

$$\begin{aligned}
\int_0^{\frac{1}{2}} \sqrt{1+4x^2}dx \quad & \overset{2x=\tan\theta}{\underset{dx=\frac{1}{2}\sec^2(\theta)d\theta}{=}} \quad \int_0^{\frac{\pi}{4}} \sqrt{1+\tan^2(\theta)}\frac{1}{2}\sec^2(\theta)\,d\theta \\
&= \quad \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec^3(\theta)\,d\theta \\
&= \quad \frac{1}{2}\left[\frac{\sec(\theta)\tan(\theta)}{2} + \frac{\ln|\sec\theta + \tan\theta|}{2}\right]_{\theta=0}^{\theta=\frac{\pi}{4}} \\
&= \quad \frac{1}{4}\left[\sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) + \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right|\right] \\
& \quad\; -\frac{1}{4}\left[\sec(0)\tan(0) + \ln|\sec 0 + \tan 0|\right] \\
&= \quad \frac{1}{4}\left[\left(\sqrt{2}\right)(1) + \ln\left|\sqrt{2}+1\right| - 0 - \ln|1|\right] \\
&= \quad \frac{\sqrt{2} + \ln\left|\sqrt{2}+1\right|}{4}
\end{aligned}$$

There is always a $y$-counterparts, that is, given a function $g(y)$ on $[c, d]$, determine the formula for the arc length. We revisit the fundamental application of Pythagorus since the length element remains the same,

$$ds(y) = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(dy)^2\left(\left(\frac{dx}{dy}\right)^2 + 1\right)} = \sqrt{1 + (g'(y))^2}dy$$

Thus,

$$S = \int_c^d \sqrt{1 + (g'(y))^2}dy.$$

# 22 Lecture XXII: Surface Area of Surfaces of Revolution

Let $f(x)$ be a nonnegative smooth function over $[a,b]$. We rotate $f(x)$ about the $x$-axis, obtaining a surface of revolution (the interior is the solid of revolution, dealt with already). The goal is to compute the surface area of the resulting surface.

Again, the idea is to figure out the surface area element by considering a small vertical strip on the surface at location $x$. At this location, we know that the arc length element is $ds(x) = \sqrt{(dx)^2 + (dy)^2}$, now representing the thickness of this strip. The thin strip, revolved around the $x$-axis, produces a band of width $dx$, and in fact, length of the circumference of the circle with radius $f(x)$, which is $2\pi f(x)$. Therefore, the surface area element satisfies

$$dA(x) = 2\pi f(x)\, ds = 2\pi f(x)\sqrt{(dx)^2 + (dy)^2} = 2\pi f(x)\sqrt{1 + (f'(x))^2}dx.$$

**Example 22.1** (Surface area of a sphere of radius $R$)**.** A sphere of radius $R$ can be obtained by rotating the curve $f(x) = \sqrt{R^2 - x^2}$ for $-R \leq x \leq R$ about the $x$-axis. Therefore, we can go ahead and use the surface area formula. We first find that

$$f'(x) = -\frac{x}{\sqrt{R^2 - x^2}}.$$

Using the formula, we write the integral carefully

$$\begin{aligned}
SA &= \int_{-R}^{R} 2\pi\sqrt{R^2 - x^2}\sqrt{1 + \left(-\frac{x}{\sqrt{R^2 - x^2}}\right)^2}dx \\
&= 2\pi \int_{-R}^{R} \sqrt{R^2 - x^2}\sqrt{1 + \frac{x^2}{R^2 - x^2}}dx \\
&= 2\pi \int_{-R}^{R} \sqrt{R^2 - x^2}\sqrt{\frac{R^2 - x^2 + x^2}{R^2 - x^2}}dx \\
&= 2\pi \int_{-R}^{R} \sqrt{R^2 - x^2}\frac{R}{\sqrt{R^2 - x^2}}dx \\
&= 2\pi R \int_{-R}^{R} dx \\
&= 2\pi R\,[R - (-R)] \\
&= 4\pi R^2
\end{aligned}$$

where the simplification is astounding.

**Example 22.2** (Surface area of a cone)**.** A cone of radius $R$ and height $H$ can be obtained by rotating the straight line $f(x) = \frac{R}{H}x$ about the $x$-axis, for $0 \leq x \leq H$. To compute the surface area, we need the derivative

$$f'(x) = \frac{R}{H}.$$

Thus,

$$\begin{aligned} SA &= \int_0^H 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx \\ &= 2\pi \int_0^H \frac{R}{H} x \sqrt{1 + \left[\frac{R}{H}\right]^2} dx \\ &= \frac{2\pi R}{H} \sqrt{1 + \left[\frac{R}{H}\right]^2} \int_0^H x dx \\ &= \frac{2\pi R}{H} \sqrt{1 + \left[\frac{R}{H}\right]^2} \frac{H^2}{2} \\ &= \pi R \sqrt{1 + \frac{R^2}{H^2}} H \\ &= \pi R \sqrt{H^2 + R^2} \end{aligned}$$

This is the surface area of a cone on the lateral. If you want to include the base, then simply add $\pi R^2$ to the answer above.

## Optional reading: An Object with Finite Volume but Infinite Surface Area

**Example 22.3** (Gabriel's horn)**.** Consider the function $f(x) = \frac{1}{x}$ on $[1, a]$ where $a \geq$ is an arbitrary number. Let $R$ be the region under $f(x)$ and above the $x$-axis.

1. Compute the volume of the solid of revolution about the $x$-axis.

2. Compute the surface area of the surface of revolution about the $x$-axis.

3. Send $a$ to $\infty$.

**Solution.** The volume of the solid is not difficult. We utilize the formula involving cross-sectional area and find

$$V(a) = \int_1^a \pi (f(x))^2 dx = \pi \int_1^a \frac{1}{x^2} dx = \pi \left[-\frac{1}{x}\right]_{x=1}^{x=a} = \pi \left[1 - \frac{1}{a}\right]$$

To use the surface area formula, we must compute the derivative

$$f'(x) = -\frac{1}{x^2}.$$

Together, we find, via some good effort of integration,

$$\begin{aligned}
SA(a) &= \int_1^a 2\pi f(x)\sqrt{1+[f'(x)]^2}dx \\
&= 2\pi\int_1^a \frac{1}{x}\sqrt{1+\left[-\frac{1}{x^2}\right]^2}dx \\
&= 2\pi\int_1^a \frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx \\
&= 2\pi\int_1^a \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}}dx \\
&= 2\pi\int_1^a \frac{1}{x}\frac{\sqrt{x^4+1}}{x^2}dx \\
&= 2\pi\int_1^a \frac{\sqrt{x^4+1}}{x^3}dx
\end{aligned}$$

$$u = x^2 \implies du = 2xdx.$$

Thus,

$$\begin{aligned}
\int_1^a \frac{\sqrt{x^4+1}}{x^3}dx &= \int_1^{a^2} \frac{\sqrt{u^2+1}}{u\sqrt{u}}\frac{du}{2\sqrt{u}} \\
&= \frac{1}{2}\int_1^{a^2}\frac{\sqrt{u^2+1}}{u^2}du
\end{aligned}$$

Here, we use the trigonometric substitution

$$u = \tan\theta \implies du = \sec^2(\theta)\,du$$

and also

$$\begin{aligned}
u = a^2 &\implies \theta = \tan^{-1}\left(a^2\right); \\
u = 1 &\implies \theta = \frac{\pi}{4}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{1}{2}\int_1^{a^2}\frac{\sqrt{u^2+1}}{u^2}du &= \frac{1}{2}\int_{\frac{\pi}{4}}^{\tan^{-1}\left(a^2\right)}\frac{\sqrt{\tan^2(\theta)+1}}{\tan^2(\theta)}\sec^2(\theta)\,du \\
&= \frac{1}{2}\int_{\frac{\pi}{4}}^{\tan^{-1}\left(a^2\right)}\frac{\sec^3(\theta)}{\tan^2(\theta)}d\theta \\
&= \frac{1}{2}\int_{\frac{\pi}{4}}^{\tan^{-1}\left(a^2\right)}\frac{1}{\cos^3(\theta)}\frac{\cos^2(\theta)}{\sin^2(\theta)}d\theta \\
&= \frac{1}{2}\int_{\frac{\pi}{4}}^{\tan^{-1}\left(a^2\right)}\sec(\theta)\csc^2(\theta)\,d\theta \\
&= \frac{1}{2}\int_{\frac{\pi}{4}}^{\tan^{-1}\left(a^2\right)}\sec(\theta)\left(\cot^2(\theta)+1\right)d\theta \\
&= \frac{1}{2}\left[\int_{\frac{\pi}{4}}^{\tan^{-1}\left(a^2\right)}\sec(\theta)\cot^2(\theta)\,d\theta+\int_{\frac{\pi}{4}}^{\tan^{-1}\left(a^2\right)}\sec(\theta)\,d\theta\right] \\
&= \frac{1}{2}\left[\int_{\frac{\pi}{4}}^{\tan^{-1}\left(a^2\right)}\frac{\cos\theta}{\sin^2(\theta)}d\theta+\ln\left|\sec\theta+\tan\theta\right|\Big|_{\theta=\frac{\pi}{4}}^{\theta=\tan^{-1}\left(a^2\right)}\right] \\
&= \frac{1}{2}\left[-\frac{1}{\sin(\theta)}+\ln\left|\sec\theta+\tan\theta\right|\right]_{\theta=\frac{\pi}{4}}^{\theta=\tan^{-1}\left(a^2\right)} \\
&= \frac{1}{2}\left[\ln\left|\sqrt{a^4+1}+a^2\right|-\frac{\sqrt{a^4+1}}{a^2}-\ln\left|\sqrt{2}+1\right|+\sqrt{2}\right]
\end{aligned}$$

Collectively,

$$\begin{aligned}
V(a) &= \pi\left[1-\frac{1}{a}\right]\to\pi,\quad\text{as } a\to\infty \\
SA(a) &= \pi\left[\ln\left|\sqrt{a^4+1}+a^2\right|-\frac{\sqrt{a^4+1}}{a^2}-\ln\left|\sqrt{2}+1\right|+\sqrt{2}\right]\to\infty,\quad\text{as } a\to\infty
\end{aligned}$$

In other words, the solid for $1\leq x<\infty$ will have a finite volume, but an infinite surface area.

# 23 Lecture XXIII: Mass and Center of Mass

## 23.1 Mass of Constant Density

**Example 23.1.** (Motivating example 1) $N$ point masses $m_1, m_2, \ldots, m_N$ sit on a weightless seesaw of length $L$. Their locations on the seesaw are described by the coordinates $0 < x_1 < x_2 < \cdots < x_N < L$. Where would you put the fulcrum so that the seesaw balances?

Suppose this fulcrum sits at $0 \le \overline{x} \le L$ and everything balances. What "everything balances" means is that the seesaw experiences zero torque, or otherwise it will tip over to the side that experiences more torque.

For the first mass, it is at a displacement $x_1 - \overline{x}$ from the fulcrum, and exerts its weight $m_1$. The amount of torque on the seesaw by the first mass is

$$\tau_1 = (x_1 - \overline{x})\, m_1.$$

Since the seesaw is in balance if the fulcrum is at $\overline{x}$, we must have total torque equal to 0, that is,

$$\sum_{i=1}^{N} \tau_i = \sum_{i=1}^{N} (x_i - \overline{x})\, m_i = 0.$$

Now, we need to solve for $\overline{x}$.

$$\sum_{i=1}^{N} (x_i - \overline{x})\, m_i = 0 \implies \sum_{i=1}^{N} x_i m_i - \sum_{i=1}^{N} \overline{x} m_i = 0 \implies \sum_{i=1}^{N} x_i m_i - \overline{x} \sum_{i=1}^{N} m_i = 0$$

which then ultimately yields

$$\overline{x} = \frac{\sum_{i=1}^{N} x_i m_i}{\sum_{i=1}^{N} m_i},$$

where we observe that the denominator is **total** mass.

We often refer to the numerator as the **first moment** of the objects, typically denoted as

$$M = \sum_{i=1}^{N} x_i m_i,$$

and the total mass

$$m = \sum_{i=1}^{N} m_i.$$

Thus, the center of mass formula can be condensed to

$$\overline{x} = \frac{M}{m}.$$

The center of mass in this case is known as the 1D center of mass, because we treat the seesaw as merely an interval of certain length. Using a similar principle, we can derive the center of mass of several objects on a plane, which is in 2D.

**Theorem 23.1** (Center of Mass of Objects in a Plane)**.** *Let $m_1, m_2, \ldots, m_n$ be point masses located in the xy-plane at points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, respectively, and let $m = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then the first moments $M_x$ and $M_y$ of the system with respect to the $x$- and $y$-axes, respectively, are given by*

$$M_x = \sum_{i=1}^{N} y_i m_i, \quad M_y = \sum_{i=1}^{N} x_i m_i.$$

*The center of mass situates at the coordinate*

$$\overline{x} = \frac{M_y}{m}, \quad \overline{y} = \frac{M_x}{m}.$$

**Example 23.2** (Find the center of mass of objects in a plane)**.** Given three points masses

$$\begin{aligned} m_1 &= 2\,\text{kg} \ @\,(-1, 3) \\ m_2 &= 6\,\text{kg} \ @\,(1, 1) \\ m_3 &= 4\,\text{kg} \ @\,(2, -2) \end{aligned}$$

Find the center of mass of the system.

**Solution.** We compute the ingredients, namely, the first moments $M_x$, $M_y$ and total mass $m$.

$$m = 2 + 6 + 4 = 12\,\text{kg}$$

$$\begin{aligned} M_y &= \sum_{i=1}^{3} m_i x_i = m_1 x_1 + m_2 x_2 + m_3 x_3 = (2)(-1) + (6)(1) + (4)(2) = -12 \\ M_x &= \sum_{i=1}^{3} m_i y_i = m_1 y_1 + m_2 y_2 + m_3 y_3 = (2)(3) + (6)(1) + (4)(-2) = 4. \end{aligned}$$

Thus,

$$\overline{x} = \frac{M_y}{m} = \frac{12}{12} = 1, \quad \overline{y} = \frac{M_x}{m} = \frac{4}{12} = \frac{1}{3}$$

We verify that the center of mass should make sense geometrically. One may connect the three points and form a triangle. The center of mass must lie in the interior of the triangle.

## 23.2 Mass and Center of Mass of A Continuous Object

So far, we have only considered point masses on a line or in a plane. The natural generalization would be objects with continuous distribution of masses, such as a continuous segment of an interval, or thin sheet of material. In the former, a continuous segment of an interval, say $[a, b]$, with density $\rho(x)$ has total mass

$$m = \int_a^b \rho(x)\, dx,$$

and first moment

$$M = \int_a^b x\rho(x)\, dx.$$

The center of mass of this line is thus $\overline{x} = \frac{M}{m}$.

**Example 23.3** (Center of mass of a Line Segment (1D)). Consider the line segment $[0, 3]$ with density $\rho(e^{-x})$. Total mass is given by

$$m = \int_a^b \rho(x)\, dx = \int_0^3 e^{-x} dx = 1 - e^{-3}.$$

The moment satisfies

$$\begin{aligned} M = \int_0^3 xe^{-x} dx &= -xe^{-x} \mid_{x=0}^{x=3} + \int_0^3 e^{-x} dx \\ &= \left[-3e^{-3} + 0\right] + \left(-e^{-x}\right)_0^3 \\ &= -3e^{-3} + \left(1 - e^{-3}\right) \\ &= 1 - 4e^{-3}. \end{aligned}$$

Thus, the center of mass is

$$\overline{x} = \frac{M}{m} = \frac{1 - 4e^{-3}}{1 - e^{-3}} \approx 0.8428.$$

This answer makes sense because the density indicates that it's denser towards 0 and less dense towards 3; the center of mass should be tilted to the denser side.

## 23.3 Mass and Center of Mass (Centroid) of a Lamina: Thin Sheets of Constant Density

For the thin sheet in 2D, we assume that the sheet is thin enough so that it can be treated as a 2D plane. Such a sheet is called a **lamina**. We further assume, in this section, that the density is constant. The geometric center of laminas is called the **centroid**. We utilize symmetry (discussed below), when applicable, to find the center of mass of lamina, in the following examples.

**Theorem 23.2** (The Symmetry Principle). *If the region $R$ is symmetric about a line $l$, then the centroid of $R$ lies on $l$.*

Now, let's analyze the center of mass of a lamina formed by the region between a function $f(x)$ and the $x$-axis, for $a \leq x \leq b$. Once again, the ingredients to calculate the center of mass, is first moments with respect to the $x$- and $y$- axes, respectively, and the total mass $m$.

Consider a vertical thin strip of width $\Delta x$ in the region $R$ at location $x_i$, and let's compute the mass of this strip. The strip has area $f(x_i)\,\Delta x$, and thus a mass of $\rho f(x_i)\,\Delta x$ where $\rho$ is the density (with units mass per length squared since we are in 2D). Therefore, the total mass of the lamina is given by the limit of a Riemann sum,

$$m \approx \sum_{i=1}^{n} \rho f(x_i)\,\Delta x \overset{n\to\infty}{\to} \rho \int_a^b f(x)\,dx.$$

This formula makes perfect sense because it is precisely the area times density.

Now, we need to figure out the first moments with respect to the $x$-axis and $y$-axis, respectively. At a particular location $x_i$, the thin strip itself has center of mass $\left(x_i, \frac{f(x_i)}{2}\right)$. The second coordinate is the curious one, but still makes sense because vertical component of the center of mass of a rectangle strip is precisely at halfway between the $x$-axis and the function value $f(x_i)$. Furthermore, the rectangle can be treated as a point mass at the center of mass, and therefore, its moment with respect to the $x$-axis follows the same calculation before for point masses. More precisely, the moment of this rectangle at $x_i$ is given by the product of the mass of the rectangle, and the distance between the center of mass and the $x$-axis,

$$M_{x_i} = \rho f(x_i)\,\Delta x \cdot \frac{f(x_i)}{2} = \rho \frac{[f(x_i)]^2}{2} \Delta x.$$

Summing over all such strips, we have

$$M_x = \sum_{i=1}^{n} \rho \frac{[f(x_i)]^2}{2} \Delta x \overset{n\to\infty}{\to} \rho \int_a^b \frac{[f(x)]^2}{2} dx.$$

The moment with respective to the $y$-axis is a little easier, since the distance from the center of mass of each strip to the $y$-axis is precisely $x_i$. Thus,

$$M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho x_i f(x_i)\,\Delta x = \rho \int_a^b x f(x)\,dx.$$

Altogether, the center of mass is given by (the same formula as previously)

$$\overline{x} = \frac{M_y}{m}, \quad \overline{y} = \frac{M_x}{m}.$$

We collect the takeaway of these derivations in the following theorem.

**Theorem 23.3** (Center of Mass of a Thin Plate in the $xy$-plane)**.** *Let $R$ denote a region bounded above by $f(x)$, below by the $x$-axis, for $a \le x \le b$. Let $\rho$ be the constant density. Then*

1. *The mass of the lamina is*

$$m = \rho \int_a^b f(x)\,dx.$$

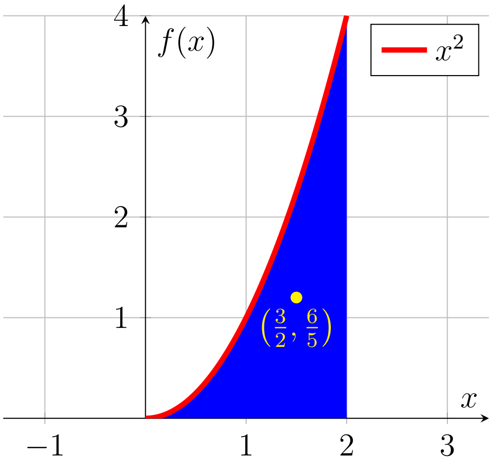
2. *The moments $M_x$ and $M_y$, with respect to the $x$- and $y$-axes, respetively, are*

$$M_x = \rho \int_a^b \frac{[f(x)]^2}{2} dx, \quad M_y = \rho \int_a^b x f(x)\,dx.$$

3. *The center of mass has coordinates*

$$(\overline{x}, \overline{y}) = \left( \frac{M_y}{m}, \frac{M_x}{m} \right).$$

**Example 23.4** (Center of Mass of Area under the Parabola)**.** Consider $f(x) = x^2$ on $[0, 2]$.




$$\begin{aligned} M_x &= \rho \int_0^2 \frac{[f(x)]^2}{2} dx \\ &= \rho \int_0^2 \frac{[x^2]^2}{2} dx \\ &= \frac{\rho}{2} \int_0^2 x^4 dx \\ &= \frac{\rho}{2} \frac{x^5}{5} \Big|_{x=0}^{x=2} \\ &= \frac{16}{5}\rho \end{aligned}$$

$$\begin{aligned} M_y &= \rho \int_0^2 x f(x)\,dx \\ &= \rho \int_0^2 x \cdot x^2 dx \\ &= \rho \int_0^2 x^3 dx \\ &= \rho \frac{x^4}{4} \Big|_{x=0}^{x=2} \\ &= 4\rho \end{aligned}$$

$$\begin{aligned} M &= \rho \int_0^2 f(x)\,dx \\ &= \rho \int_0^2 x^2 dx \\ &= \rho \frac{x^3}{3} \Big|_{x=0}^{x=2} \\ &= \frac{8}{3}\rho \end{aligned}$$

Thus,

$$\overline{x} = \frac{M_y}{M} = \frac{4\rho}{\frac{8}{3}\rho} = \frac{3}{2}$$

$$\overline{y} = \frac{M_x}{M} = \frac{\frac{16}{5}\rho}{\frac{8}{3}\rho} = \frac{6}{5}$$

**Question 23.1.** Now, based on the last example, consider the region between $f(x) = x^2$ and $g(x) = -x^2$ on $[0, 2]$. Can you confirm that the center of mass is at $\left(\frac{3}{2}, 0\right)$, by utilizing to the symmetry of the region?