MATH 1021 Lecture I
Functions, Notations and the Big Picture of Calculus

August 22, 2024

1 Function Overview

A function represents a relationship between **two** sets (collections of objects); furthermore, this relationship has the property that each input is assigned to exactly one output. In high school, this property can be tested by the "vertical line test" that checks whether a curve in the xy-plane is the graph of a function.

Functions are used in various aspects of natural sciences, engineering, economics and more frequently, mathematics. The population of cells in a petri dish, the trajectory of a comet and the price of a stock, among many other curiosities, can be represented by functions. However, there are several questions you should think about.

1. What are the corresponding "two sets" in these examples? [See Section 2 below]
2. How are these functions obtained/found? [See Section 3 below]
3. What can you do with them? [See Section 4 below]

Of course, let's also stay up to date with current events – Artificial Intelligence. What exactly is achieved by AI "gadgets"? For example, self-driving cars. The cameras/sensors take in (real-time) environment data as **inputs**, "process them" (we will talk more about what exactly is done), "finds" the function(s) that **outputs** one or a sequence of actions. In other words, AI generates instructions for itself. This very procedure relies heavily on the notion of functions. At the end of the day, AI is just another word for function(s)-find, and you better find them fast.

Functions are tools to model and predict real scientific phenomena. Their uses are by no means exact replica of the real-world, but they are the best thing we can get our hands onto that can utilize basic mathematical operations. It is useful to learn their individual exact properties, and then their merits and deficiencies in approximating reality.

2 Basic Notations for Sets and Functions

2.1 Domain and Range

A **function** f consists of a set of inputs (**the domain**, $Dom(f)$ or $D(f)$), a set of outputs (**the range** $Rg(f)$ or $R(f)$), and **a rule** for assigning (mapping) each input to exactly one output. The rule consists of, often, basic algebraic operations such as addition, subtraction, multiplication and division, and also more complicated **nonlinear** operations such as powers, exponential, trigonometric and logarithmic. We explore more of these operations in a later section.

We typically write

$$f : X \to Y$$

that reads, "f maps from the set X to the set Y". The domain depends on your choice; the range depends on the domain and also properties of the rule.

Example. Consider the function f where the domain is the set of all real numbers, $\mathbb{R}$ (blackboard R), and the rule is to square the input.

Then, clearly the input $x = 3$ is mapped to the value $3^2 = 9$. Notice that every real number squared yields a non-negative real number, the range of this function is the set of all non-negative real numbers. In mathematical notations, we write $y = f(x) = x^2$, where x is the **independent variable** and y is the **dependent variable**.

Example. Here is a function whose domain depends your choice, and in particular, what you want as outputs.

$$y = f(x) = \sqrt{x}.$$

If we want outputs as real numbers, then it makes no sense to **choose** the domain as the set of all real numbers, because for example, for a negative input $x = -1$, we have $y = \sqrt{-1} = i$, which is NOT a real number output (known as the imaginary unit). Therefore, if one asks you "what's the domain of this square root function?", you should really ask, "what do you want the output to be?", or like a pro "outputting reals?" before answering. Typically, in first year calculus, we work strictly with **real numbers**, and therefore, we say the **natural domain** of the square root function is the set of non-negative real numbers, expecting the output to be also non-negative real numbers. The question we ask you in quizzes or exams about domains is really asking for the **natural domain** of a function.

2.2 Interval Notation

We are familiar with the real number line (the real line in short). To describe a set of numbers, we often use **the interval notation**. For example, for real numbers strictly between 1 and 3, we may write the set

$$\{x \mid 1 < x < 3\},$$

read as "the set of real numbers that lies strictly between 1 and 3". Equivalently, we also write $(1, 3)$, with brackets to indicate the strict inequality. You may use

$$(1, 3) = \{x \mid 1 < x < 3\}$$

interchangeably. Furthermore, $(1, 3)$ is an example of **an open interval**. For a closed interval, consider the endpoints being included, namely,

$$[1, 3] = \{x \mid 1 \leq x \leq 3\}.$$

The set of non-negative real numbers is written as

$$[0, \infty) = \{x \mid x \geq 0\},$$

with the understanding that ∞ is NOT a real number (but a symbol/concept). Similarly, the set of non-positive numbers is

$$(-\infty, 0] = \{x \mid x \leq 0\}.$$

The entire real line is written as

$$(-\infty, \infty) = \{x \mid -\infty < x < \infty\} = \{x \mid x \text{ is any real number } 0\}.$$

2.3 Logical Expression

That x is an element of the open interval (a, b) can be shortened into the following mathematical expression

$$x \in (a, b)$$

where the symbol $\in$ represents the English words "is an element of/belongs to".

A subset A of a larger set B is written as $A \subseteq B$ (A may be exactly equal to B or smaller), noting that $\subset$ and $=$ parts are strict subsets and set equality. An example would be

$$(1, 2) \subset [1, 2]$$

since the closed interval $[1, 2]$ contains every point in the open interval $(1, 2)$. Another example would be

$$(3, \infty) \subset (2, \infty)$$

because numbers larger than 3 must necessarily satisfy that they are larger than 2.

Given two real numbers a, b, there are only three possible outcomes of a comparison between them,

$$a < b, \quad a = b, \quad a > b.$$

Between two mathematical statements, we use the symbol $\Longrightarrow$ to replace the English word "implies". For example, the true implication "$x > 3$ implies $x > 2$" is written as

$$x > 3 \implies x > 2.$$

As mathematicians, we often examine whether an implication, a basic element of a mathematiacl proof, is true or false.

There are great interplays between set inclusion and implications, which we do not detail here. They are more discussed in Discrete Mathematics or an Introduction to Mathematical Logic.

2.4 Examples

Express the domain and range of the following functions in interval notations.

1. $f(x) = x^2 + 1$.

$$Dom(f) = (-\infty, \infty),$$
$$R(f) = [1, \infty).$$

2. $g(x) = \sqrt{3x+2} - 1$

$$Dom(g) = [-\frac{2}{3}, \infty),$$

For the range, notice that
$$y = g(x) = \sqrt{3x+2} - 1 \geq -1$$
because $\sqrt{3x+2} \geq 0$. Therefore, $\{y \mid y \geq -1\} \subseteq R(g)$, that is, the observed set of possible outputs should be a **subset** of the true range. Here, to verify that $\{y \mid y \geq -1\}$ is truly the full range is an intricate procedure based on the **definition** of domain and range. We need to show that for every output y such that $y \geq -1$, we must associate it with a real number $x \in Dom(f)$ such that $f(x) = y$. We find that x satisfies the equation

$$\sqrt{3x+2} - 1 = y \implies \sqrt{3x+2} = y + 1 \implies 3x + 2 = (y+1)^2 \implies x = \frac{(y+1)^2}{3} - \frac{2}{3}.$$

The question now is whether this x belongs to the domain $Dom(g) = [-\frac{2}{3}, \infty)$ we just claimed. Indeed, we note that
$$x = \frac{(y+1)^2}{3} - \frac{2}{3} \geq -\frac{2}{3}$$
which is precisely the domain we claimed. Therefore, $[-1, \infty)$ is the true range of $R(f)$, corresponding to the natural domain $[-\frac{2}{3}, \infty)$.

3 The Cartesian Coordinate System and the Graph of a Function

The definition of a function given so far is unfortunately quite abstract. We are visual animals, on the other hand. It makes sense then to demonstrate the relationship claimed by a function $y = f(x)$ by virtue of seeing things "high" and "low". What do we mean by "high" and "low"? We need a metric, namely, a ruler.

Introduce the xy-plane.

3.1 Given $y = f(x)$, graph it.

The chore here is to make a table. You choose the x-values and then calculate $f(x)$.

Example. $f(x) = \sqrt{x}$.

Example. $g(x) = x^2$.

3.2 Given data, fit a function $y = f(x)$ to it.

In reality, instead of being handed a function, we have data provided – namely, a table in the same format as in the last section. In other words, we are provided points on a curve, and we seek a function whose graph may pass through all of them. This function then may be used to further predict the behaviour of a physical system.

Unfortunately, if you are provided three points, there is a very high chance that even a line can't even go through them all at once. Then, how about a quadratic? How about a cubic? How about a polynomial? Or, with more seemingly insane thoughts, sines and cosines?

These topics belong to the study of statistics, more precisely, the study of **regression**. In **linear regression**, A line through a bunch of points certainly misses every point, but one should believe that there is such an optimal line such that the error of approximation is minimized. Guess what? This problem is after all an **optimization problem**, which we will end up learning in this term.

3.3 Function Transformations

Transformation of f for $c > 0$	Effect on the graph of f	Nomenclature
$f(x) \pm c$	Vertical shift up/down c units	Vertical Translation
$f(x \pm c)$	Horizontal shift left/right by c units	Horizontal Translation
$cf(x)$	Vertical expansion if $c > 1$, compression if $0 < c < 1$.	Vertical Expansion/Compression
$f(cx)$	Horizontal expansion if $0 < c < 1$, compression if $c > 1$.	Horizontal Expansion/Compression
$-f(x)$	Reflection about the x-axis	Reflection
$f(-x)$	Reflection about the y-axis	Reflection

A table is handy, but brute-force memorization defeats the purpose of learning. See HW1 for an exercise.

4 Classes of Functions

4.1 Polynomials

Let $a_0, a_1, a_2, \dots, a_n$ be real numbers. Then, a polynomial of degree n is defined as

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n.$$

The a_i's are called the **coefficients** of the polynomial, and they completely characterize the polynomial, i.e., knowing them determines the polynomial. Therefore, the determination of a polynomial of degree n requires $(n+1)$ equations because we have $(n+1)$ unknown coefficients we need to solve for.

A line then is simply a polynomial of degree 1, and a constant is a polynomial of degree 0,

$$y(x) = a_0 + a_1 x$$

or in point-slope form

$$y - y_0 = m(x - x_0).$$

Once again, we need two equations/conditions so we can find a_0 and a_1. In the point-slope form, we are typically provided two points (x_0, y_0) and (x_1, y_1) so that $m = \frac{y_1 - y_0}{x_1 - x_0}$, the classical rise-over-run.

4.2 Trigonometric

Function Name	Symbol	Natural Domain	Range	Typical Period	Relationship
Sine	$\sin(x)$	$(-\infty, \infty)$	$[-1, 1]$	$[0, 2\pi]$	Fundamental
Cosine	$\cos(x)$	$(-\infty, \infty)$	$[-1, 1]$	$[0, 2\pi]$	Fundamental
Tangent	$\tan(x)$	$(-\infty, \infty) \setminus \left\{(2n+1)\frac{\pi}{2}\right\}$ for n an integer	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\tan(x) = \frac{\sin(x)}{\cos(x)}$
Cotangent	$\cot(x)$	$(-\infty, \infty) \setminus \{n\pi\}$ for n an integer	$(-\infty, \infty)$	$(0, \pi)$	$\cot(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)}$
Cosecant	$\csc(x)$	$(-\infty, \infty) \setminus \{n\pi\}$ for n an integer	$(-\infty, -1] \cup [1, \infty)$	$(-\pi, \pi) \setminus \{0\}$	$\csc(x) = \frac{1}{\sin(x)}$
Secant	$\cos(x)$	$(-\infty, \infty) \setminus \left\{(2n+1)\frac{\pi}{2}\right\}$ for n an integer	$(-\infty, -1] \cup [1, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\sec(x) = \frac{1}{\cos(x)}$

Once again, a table is handy, but brute-force memorization defeats the purpose of learning. See HW1 for an exercise.

MATH 1021 Lecture II
Limits, its Motivations and its Intuitive Definition from Graphs of Functions

August 23, 2024

1 Motivation

What would be a cost-friendly way to prove speeding on the highway? Say, you set up two cameras/detectors 2 miles apart, at location s_0 and s_1 where $s_1 - s_0 = 2$. For each car, the cameras register the time when the car passes by them, say t_0 and t_1. Then, the average speed of the vehicle, across this time interval $[t_0, t_1]$, is given by the distance it has traveled divided by the time used,

$$v_0^{(\mathrm{ave})} = \frac{s_1 - s_0}{t_1 - t_0}.$$

Now, if $v_{\mathrm{ave}} > v_{\mathrm{limit}}$, then the driver definitely has sped.

What if $v_{\mathrm{ave}} < v_{\mathrm{limit}}$? Does it guarantee that the driver hasn't sped? No, not necessarily. The driver might have sped for a little and started driving well below the speed limit, and ended up averaging below the speed limit. How can we then prove whether the driver still has sped? One possible way is to put more cameras, i.e., increase camera density so that the distance between checkpoints are shortened. In fact, one may set up so many cameras that the data (the quotient above) nearly reproduces the history of the car's speedometer. In other words, from gauging the distance traveled across various time intervals help us decipher the finer details of speed. In fact, we will be able to construct a table of numbers that list the history of average speeds.

	t	$\Delta s_j = s_{j+1} - s_j$	$v_j^{(\mathrm{ave})}$ over $[t_j, t_{j+1}]$
t_0	0	1	$\frac{1}{1.23}$
t_1	1.23	1	$\frac{1}{2.54-1.23}$
t_2	2.54	1	$\frac{1}{3.44-2.54}$
t_3	3.44	1	$\frac{1}{4.16-3.44}$
t_4	4.16	/	/

The list of average speeds will improve to the list of **instantaneous speed** if the registered times are sufficiently close, i.e., so small that they are beyond our natural perception. Provided distance and time, the act of finding **instantaneous speed** is via differentiation (Calculus I). Meanwhile, we can reverse the process – provided speed and time, the act of finding **distance** is via integration (Calculus II).

2 Limits and its Intuitive Definition from Graphs of Functions

So, it seems that we need to shrink the time interval from distance data to gauge the distance changed, and incidentally, speed. Another way of "shrinking" is by considering the first time stamp fixed, and you are bringing the larger one closer and closer to it. Meanwhile, the distance data is still read according to the shrinking time. We posit that distance is a function of time,

$$s = f(t).$$

Definition. Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. If **all** values of the function $f(x)$ approach the same real number L as the values of x $(\neq a)$ approach the number a, then we say that the limit of $f(x)$ as x approaches a is L.

Too long to remember? Here is a concise version:

As x gets closer to a, $f(x)$ gets closer **and stays close** to L. We write

$$\lim_{x\to a} f(x) = L.$$

The example where the function has a jump is the case where the limit does NOT exist, because $f(x)$ did NOT stay close to L (whichever L you choose on whichever side).

More computationally, we examine the limit of a function near $x = a$ by considering input values close to a and the corresponding function values. We make a table, as always. We approach a from below

x	$f(x)$
$a - 10^{-1}$	$f\left(a - 10^{-1}\right)$
$a - 10^{-2}$	$f\left(a - 10^{-2}\right)$
$a - 10^{-3}$	$f\left(a - 10^{-3}\right)$
$a - 10^{-4}$	$f\left(a - 10^{-4}\right)$

and from above

x	$f(x)$
$a + 10^{-1}$	$f\left(a + 10^{-1}\right)$
$a + 10^{-2}$	$f\left(a + 10^{-2}\right)$
$a + 10^{-3}$	$f\left(a + 10^{-3}\right)$
$a + 10^{-4}$	$f\left(a + 10^{-4}\right)$

Example. $f(x) = \frac{x^2-4}{x-2}$.

The only "bad point" is $x = 2$ because direct evaluation yields division by 0. However, we can still evaluate a limit towards $x = 2$, without arriving at the point $x = 2$.

x	$\frac{x^2-4}{x-2}$
$2 - 10^{-1}$	3.9000
$2 - 10^{-2}$	3.9900
$2 - 10^{-3}$	3.9990
$2 - 10^{-4}$	3.9999

x	$f(x)$
$2 + 10^{-1}$	4.1000
$2 + 10^{-2}$	4.0100
$2 + 10^{-3}$	4.0010
$2 + 10^{-4}$	4.0001

The limit is very clear that

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = 4.$$

Now, should we always rely on a table? It seems awfully inefficient. Let's focus on what the limit sign allows us to do. Note that since we are taking a limit as x approaches 2, we are NOT exactly at $x = 2$. Therefore, we can still try to get rid of $x - 2$ by factoring the numerator,

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2} (x+2) = 4.$$

This is a prime example where the limit sign helps us simplify the expression. It is thus also important that you continue writing the limit sign until you evaluate it at the last step.

2.1 Simple Limits

Theorem. *Let a be a real number and c be a constant.*

$$\lim_{x\to a} x = a,$$

$$\lim_{x\to a} c = c.$$

2.2 Existence of a Limit

The table we made above seems to suggest that we may approach a single point $x = a$ from two directions – above and below. It is important that the function values, from **both directions**, stay close to the limiting value L. What happens if they don't?

Consider the function

$$f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

The limit from the left side is -1 while the limit from the right is 1. There is a mismatch, which defeats the intuitive definition that the function value much stay close to a common limiting value L. In this case, we say **the limit does not exist**. More precisely, we ought to have a notion of one-sided limits.

Definition. [One-sided limit]

Limit from the left

Let $f(x)$ be a function defined at all values in an open interval of the form (c, a), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x < a$) approach number a, then we say that L is the limit of $f(x)$ as x approaches a from the left. We write

$$\lim_{x \to a^-} f(x) = L.$$

Limit from the right

Similarly, flip all $<$ to $>$, we have the limit from the right,

$$\lim_{x \to a^+} f(x) = L.$$

Theorem. *Let $f(x)$ be a function defined at all values in an opoen interval containing a, with the possible exception of a itself, and let L be a real number. Then*

$$\lim_{x \to a} f(x) = L$$

IF AND ONLY IF

$$\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$

Remark. "If and only if", symbolically, $\iff$, means that the two statements imply each other. If you establish one side, you automatically obtain the other side. It is the strongest form of implication, and certainly the most restrictive.

Example. Consider the piecewise function

$$f(x) = \begin{cases} x + 1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases}.$$

Find the following one-sided limits.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x + 1) = 3,$$
$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \left(x^2 - 4\right) = 0.$$

Here, the function achieves different one-sided limits, and we say that the limit does NOT exist at $x = 2$.

MATH 1021 Lecture III
Limit Laws and The Squeeze Theorem

August 25, 2024

After class reading: Discussion 1, and Examples 2.21, 2.22, 2.23 (page 167-170) in the main text.

1 Limit Evaluation Techniques [Section 2.3]

Last week, we learned

$$\lim_{x\to a} x = a$$
$$\lim_{x\to a} c = c$$

which are the two fundamental limits. We refer to them as the fundamental laws, labeled as "F".

1.1 Limit Laws

Limits, if exist, are just real numbers. So, they follow the same operation rules as real numbers.

Theorem. *[Limit Laws] Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a (a good environment for the point of interest). Assume that L and M are real numbers such that*

$$\lim_{x\to a} f(x) = L, \quad \lim_{x\to a} g(x) = M.$$

Then,

1. $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$. *[Sum and Difference can split].*
2. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x) = cL$, *for c a constant.*
3. $\lim_{x\to a} f(x)g(x) = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x)) = LM$. *[A Product splits - but only when each individual limit exists].*
4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$ *for $M \neq 0$. [A Quotient splits – but only when each individual limit exists, and $M \neq 0$].*
5. $\lim_{x\to a}(f(x))^n = (\lim_{x\to a} f(x))^n = L^n$, *for every positive integer n.*
6. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L} = L^{\frac{1}{n}}$ *for all L if n is odd; and for all $L \geq 0$ if n is even and $f(x) \geq 0$. [Think about why the split between odd and even n].*

Example. Find $\lim_{x\to -3}(4x+2)$.

In the following, we refer to each equality with the number of the laws called. Pay close attention now and soon this procedure will be very quick.

$$\lim_{x\to -3}(4x+2) \overset{(1)}{=} \lim_{x\to -3} 4x + \lim_{x\to -3} 2 \overset{(2)}{=} 4\lim_{x\to -3} x + \lim_{x\to -3} 2 \overset{(\mathrm{F})}{=} 4(-3) + 2 = -12 + 2 = -10.$$

Example. Find $\lim_{x\to 2}\frac{2x^2-3x+1}{x^3+4}$.

Again, we follow the limit laws.

$$\begin{aligned}
\lim_{x\to 2}\frac{2x^2-3x+1}{x^3+4} &\overset{(4),\text{ and }\left(2^3\right)+4=12\neq 0}{=} \frac{\lim_{x\to 2}2x^2-3x+1}{\lim_{x\to 2}x^3+4}\\
&\overset{(1)}{=} \frac{\lim_{x\to 2}2x^2-\lim_{x\to 2}3x+\lim_{x\to 2}1}{\lim_{x\to 2}x^3+\lim_{x\to 2}4}\\
&\overset{(2)}{=} \frac{2\lim_{x\to 2}x^2-3\lim_{x\to 2}x+\lim_{x\to 2}1}{\lim_{x\to 2}x^3+\lim_{x\to 2}4}\\
&\overset{(F)}{=} \frac{2\left(2\right)^2-3\cdot 2+1}{2^3+4}\\
&= \frac{1}{4}.
\end{aligned}$$

Now, do we must follow the limit laws every single time? It does give correct results, but it seems awfully slow to get to the correct result. Is there a nice theorem for limits of polynomials and rational functions?

Theorem. *[Limits of Polynomial and Rational Functions] Let $p(x)$ and $q(x)$ be polynomial functions. Let a be a real number. Then*

$$\lim_{x\to a}p(x)=p(a)$$

and

$$\lim_{x\to a}\frac{p(x)}{q(x)}=\frac{p(a)}{q(a)},\quad q(a)\neq 0.$$

Proof. Sketch of proof. Write

$$p(x)=c_nx^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0.$$

Then

$$\lim_{x\to a}p(x)=\lim_{x\to a}\left(c_nx^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0\right).$$

Apply the limit law (1) (for the sum), (2) (for the constant coefficients) and (5) (for the powers). A similar result holds for $\lim_{x\to a}q(x)$. The second statement in the Theorem follows by limit law (4). □

An additional example for you to read.

Example. Find $\lim_{x\to 6}(2x-1)\sqrt{x+4}$.

$$\begin{aligned}
\lim_{x\to 6}(2x-1)\sqrt{x+4} &\overset{(3)}{=} \left(\lim_{x\to 6}(2x-1)\right)\left(\lim_{x\to 6}\sqrt{x+4}\right)\\
&\overset{(1)+(6)}{=} \left(\lim_{x\to 6}2x-\lim_{x\to 6}1\right)\left(\sqrt{\lim_{x\to 6}(x+4)}\right)\\
&\overset{(2)+(1)}{=} \left(2\lim_{x\to 6}x-\lim_{x\to 6}1\right)\left(\sqrt{\lim_{x\to 6}x+\lim_{x\to 6}4}\right)\\
&\overset{(F)}{=} (2\cdot 6-1)\left(\sqrt{6+4}\right)\\
&= 11\sqrt{10}.
\end{aligned}$$

Remark. Please don't give me $\sqrt{x+4}=\sqrt{x}+\sqrt{4}$.

Remark. If you can do it slowly, you can do it quickly.

1.2 Factorization

We saw the factorization technique earlier in limits such as

$$\lim_{x\to 2}\frac{x^2-4}{x-2},$$

thanks to having the limit sign here which makes sure that x is NOT exactly 2.

Now, for a general problem

$$\lim_{x\to a}\frac{p(x)}{q(x)},$$

one may appeal to the theorem in Section 1.1. However, the case that wasn't covered was $q(a)=0$. Meanwhile, if we also have $q(a)=0$, direct evaluation of the limit yields the **indeterminate form** "$\frac{0}{0}$". We must be more careful (you will also encounter this in Calculus II with another technique).

Now, if $p(a)=0$ and $q(a)=0$, then $x=a$ is necessarily a root of both polynomials $p(x)$ and $q(x)$. It makes sense that we should factor both $p(x)$ and $q(x)$ and try to extract the factor $(x-a)$.

Example. Evaluate $\lim_{x\to 3}\frac{x^2-3x}{2x^2-5x-3}$.

1. Direct evaluation yields "$\frac{0}{0}$". We need to factor.

2.
$$\begin{aligned}\lim_{x\to 3}\frac{x^2-3x}{2x^2-5x-3} &= \lim_{x\to 3}\frac{x(x-3)}{(2x+1)(x-3)}\\ &= \lim_{x\to 3}\frac{x}{2x+1}\\ &= \frac{3}{7}.\end{aligned}$$

1.3 Conjugation

There are other situations where we run into "$\frac{0}{0}$".

Example. Evaluate $\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}$.

1. Direct evaluation yields "$\frac{0}{0}$". But the numerator isn't a polynomial, so factoring might be difficult.

2. Apply **the method of rationalization**, also known as, **multiply and divide by the conjugate of the radical expression**.

$$\begin{aligned}\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1} &= \lim_{x\to -1}\frac{\left(\sqrt{x+2}-1\right)\left(\sqrt{x+2}+1\right)}{(x+1)\left(\sqrt{x+2}+1\right)}\\ &= \lim_{x\to -1}\frac{(x+2)-1}{(x+1)\left(\sqrt{x+2}+1\right)}\\ &= \lim_{x\to -1}\frac{(x+1)}{(x+1)\left(\sqrt{x+2}+1\right)}\\ &= \lim_{x\to -1}\frac{1}{\sqrt{x+2}+1}\\ &= \frac{1}{\sqrt{-1+2}+1}\\ &= \frac{1}{2}.\end{aligned}$$

Remark. Please, put parenthesis wherever appropriate, namely, at places if a parenthesis isn't there, the expression changes meaning. For example, at the first step above,

$$\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1} = \lim_{x\to -1}\frac{\left(\sqrt{x+2}-1\right)\left(\sqrt{x+2}+1\right)}{\boxed{(x+1)}\left(\sqrt{x+2}+1\right)}.$$

Don't write

$$\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1} = \lim_{x\to -1}\frac{\left(\sqrt{x+2}-1\right)\left(\sqrt{x+2}+1\right)}{x+1\left(\sqrt{x+2}+1\right)}$$

because the denominator now is equal to $x+\sqrt{x+2}+1$, which is utter garbage.

1.4 More Complex Fractions

Example. $\lim_{x\to 1}\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$.

1. Direct evaluation yields "$\frac{0}{0}$". But the numerator is quite some ugly non-polynomial.
2. General rule when something is complicated – simplify it. Here, combine the fractions

$$\begin{aligned}\lim_{x\to 1}\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1} &= \lim_{x\to 1}\frac{\frac{2-(x+1)}{2(x+1)}}{x-1}\\ &= \lim_{x\to 1}\frac{2-x-1}{(x-1)\,2\,(x+1)}\\ &= \lim_{x\to 1}\frac{1-x}{(x-1)\,2\,(x+1)}\\ &= -\lim_{x\to 1}\frac{1}{2\,(x+1)}\\ &= -\frac{1}{4}.\end{aligned}$$

Try the following yourself.

$$\lim_{x\to 3}\left(\frac{1}{x-3}-\frac{4}{x^2-2x-3}\right).$$

1.5 Limits for Trigonometric Functions

We have fully dealt with polynomial and rational functions. Now, we move onto limits with trigonometric functions.

$$\lim_{\theta\to 0^-}\sin\theta = 0 = \lim_{\theta\to 0^+}\sin\theta$$

by looking at the unit circle. Then, by Pythagoras and limit law (6) + (5), we have

$$\lim_{\theta\to 0}\cos\theta = \lim_{\theta\to 0}\sqrt{1-\sin^2\theta} = 1.$$

1.6 The Squeeze Theorem

Let's start with a seemingly impossible problem

$$\lim_{x\to 0} x^4\cos\frac{1}{x}.$$

One thing you should be aware of is that $\lim_{x\to 0}\cos\left(\frac{1}{x}\right)$ does NOT exist because the closer x gets to 0, the faster the function oscillates, and never settles close near a value. So, the product rule won't help here.

What if we can use basic algebra and properties of cosine to show that the function can be enveloped between two other better looking functions? Furthermore, these two better looking functions have a common limit as $x\to 0$. This will squeeze the function of interest into that common limit as well. This procedure is called **the Squeeze Theorem**, stated in more detail below.

Theorem. *[The Squeeze Theorem] Let $f(x)$, $g(x)$ and $h(x)$ be defined for all $x\neq a$ over an open interval containing a. If*

$$f(x)\le g(x)\le h(x)$$

for all $x\neq a$ in an open interval containing a, and

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$

where L is a real number, then $\lim_{x\to a} g(x) = L$.

With the Squeeze Theorem, we can now tackle the original problem.

Example.

$$\lim_{x\to 0} x^4 \cos\frac{1}{x}.$$

First, note that

$$-1 \le \cos\left(\frac{1}{x}\right) \le 1$$

for every real number x except $x = 0$. Therefore, multiplying the positive number x^4 (being positive is important) to both sides of the inequality, we have

$$-x^4 \le x^4 \cos\left(\frac{1}{x}\right) \le x^4.$$

Here, we have found both sides of the envelope, namely, $f(x) = -x^4$ and $h(x) = x^4$. Their limits as x goes to 0 are very simple,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(-x^4\right) = 0,$$
$$\lim_{x\to 0} h(x) = \lim_{x\to 0} x^4 = 0.$$

Thus, by the Squeeze theorem, the middle function must go to the same limit, i.e.,

$$\lim_{x\to 0} x^4 \cos\frac{1}{x} = 0.$$

So, what if now we face yet again, the case of "$\frac{0}{0}$", in the setting of trigonometric functions?

Example. $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$.

We need a LOT of help from a visual. See Figure 2.29 on Page 172.

The biggest realization is a geometric one. We find that for $0 \le \theta \le \pi/2$,

$$\sin\theta \le \theta \le \tan\theta.$$

Then, dividing both sides by the positive quantity $\sin\theta$ (positive because θ is restricted to $\left[0, \frac{\pi}{2}\right]$), we have

$$1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}.$$

Now, flipping the inequalities, we have

$$\cos\theta \le \frac{\sin\theta}{\theta} \le 1.$$

We note that $\lim_{\theta\to 0}\cos(\theta) = 1$ and $\lim_{\theta\to 0} 1 = 1$. Thus, by the Squeeze Theorem,

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

1.7 FREEDOM

Equipped with all of the above, you are now FREE to manipulate anything in any permissible way. You should try to massage the expression into a position that familiar limits show up, such as the last example.

Example.

$$
\begin{aligned}
\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} &= \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}\frac{1+\cos\theta}{1+\cos\theta} \\
&= \lim_{\theta\to 0}\frac{1-\cos^2\theta}{\theta\,(1+\cos\theta)} \\
&= \lim_{\theta\to 0}\frac{\sin^2\theta}{\theta\,(1+\cos\theta)} \\
&= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\frac{\sin\theta}{(1+\cos\theta)} \\
&= \left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)\left(\lim_{\theta\to 0}\frac{\sin\theta}{(1+\cos\theta)}\right) \\
&= (1)\left(\frac{0}{1+1}\right) \\
&= 0
\end{aligned}
$$

MATH 1021 Lecture IV
Infinite Limits, Limits at Infinity and Asymptotes

August 29, 2024

After lecture reading: page 425 Guidelines for Drawing the Graph of a Function, Examples 4.28-4.31.

So far, we have covered limit laws and techniques for "$\frac{0}{0}$" situations under the setting of limits of rational functions. These are plenty tools for the job. It is now time to tackle the notion of infinity, an unexplored world of excitement.

1 Infinite Limits

First, a motivating example. Consider the function

$$f(x) = \frac{1}{x}.$$

As $x \to 0^+$, the function value $\frac{1}{x}$ is positive and getting larger and larger. In fact,

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty.$$

Meanwhile, as $x \to 0^-$, the function value $\frac{1}{x}$ is negative and getting larger and larger in magnitude. We have

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

Certainly, from this, we can say that the full limit

$$\lim_{x \to 0} \frac{1}{x} \text{ does not exist.}$$

But the left and right limits tending to infinities warrant a proper definition.

Definition. [Infinite Limits]

1. Infinite limits from the left. Let $f(x)$ be a function defined at all values in an open interval of the form (b, a).
   (a) If the values of $f(x)$ **increase** without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is **positive** infinity and we write
   $$\lim_{x \to a^-} f(x) = +\infty.$$
   (b) ... **decrease** ... **negative** infinity and we write
   $$\lim_{x \to a^-} f(x) = -\infty.$$
2. Infinite limits from the right. Let $f(x)$ be a function defined at all values in an open interval of the form (a, c).

(a) If the values of $f(x)$ **increase** without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the left is **positive** infinity and we write

$$\lim_{x \to a^+} f(x) = +\infty.$$

(b) ... **decrease** ... **negative** infinity and we write

$$\lim_{x \to a^+} f(x) = -\infty.$$

3. Two-sided infinite limits. Let $f(x)$ be a function defined for all $x \neq a$ in an open interval containing a.

(a) If the values of $f(x)$ **increase** without bound as the values of x (where $x \neq a$) approach the number a, then we say the limit as x approaches a is **positive** infinity and we write

$$\lim_{x \to a} f(x) = +\infty.$$

(b) ... **decrease** ... **negative** infinity and we write

$$\lim_{x \to a} f(x) = -\infty.$$

Example. The function $f(x) = \frac{1}{x^2}$ has a well-defined limit as $x \to 0$. Due to the square, approaching from the left or right to 0 yields positive function values. These values are increasing without bounds as $x \to 0$. Thus,

$$\lim_{x \to 0} \frac{1}{x^2} = +\infty.$$

So, it seems that the oddness or evenness of the power plays a role in determining whether a limit exists when the denominator goes to 0 as $x \to a$. The following theorem collects these results.

Theorem. *If n is a positive even integer, then*

$$\lim_{x \to a} \frac{1}{(x-a)^n} = +\infty.$$

If n is a positive odd integer, then

$$\lim_{x \to a^+} \frac{1}{(x-a)^n} = +\infty,$$

and

$$\lim_{x \to a^-} \frac{1}{(x-a)^n} = -\infty.$$

Remark. This Theorem helps us tremendously in getting the graphs of functions of these reciprocal forms.

We often say "blowing up" when a function achieves an infinite limit as $x \to a$. When a is a finite real number, the vertical line $x = a$ is refered to as a vertical asymptotes of $f(x)$. More precisely,

Definition. Let $f(x)$ be a function. If any of the following conditions hold, then the line $x = a$ is a **vertical asymptote** of $f(x)$.

$$\lim_{x \to a^-} f(x) = \pm\infty,$$
$$\lim_{x \to a^+} f(x) = \pm\infty,$$

OR

$$\lim_{x \to a} f(x) = \pm\infty.$$

Example. Find limit of the function $f(x) = \frac{1}{(x+3)^4}$ as $x \to -3$.

Solution. We readily identify that the power is even. So,

$$\lim_{x \to -3} \frac{1}{(x+3)^4} = +\infty.$$

See the plots we did in class.

Now, can a function have more than one vertical asymptotes? Yes, in fact, it can have as many as the number of zeros in the denominator.

Example. Find all vertical asymptotes of the function $f(x) = \frac{1}{x^2-5x+6}$ and graph it.

Solution. We see that

$$f(x) = \frac{1}{x^2-5x+6} = \frac{1}{(x-2)(x-3)}.$$

So, the vertical asymptotes are at $x = 2$ and $x = 3$.

The question now is about how the curves bend. We tackle the left and right limits at each vertical asymptotes one-by-one.

1. Near $x = 2$,

$$\lim_{x\to 2^+} \frac{1}{(x-2)(x-3)} = -\infty$$

because as x approach 2 from the right, $x-2 > 0$ but $x-3 < 0$, so opposite signs gives us negative. However,

$$\lim_{x\to 2^-} \frac{1}{(x-2)(x-3)} = +\infty.$$

2. Near $x = 3$,

$$\lim_{x\to 3^+} \frac{1}{(x-2)(x-3)} = +\infty$$

and

$$\lim_{x\to 3^-} \frac{1}{(x-2)(x-3)} = -\infty.$$

Collecting these results, we now graph the function (see the one done in class).

2 Limit at Positive and Negative Infinity

Definition. [Limit at Infinity]

If the values of $f(x)$ becomes arbitrarily close to L as x becomes sufficiently large, we say the function f has a **limit at infinity** and write

$$\lim_{x\to\infty} f(x) = L.$$

Similarly, if the values of $f(x)$ becomes arbitrarily cloes to L for $x < 0$ as $|x|$ becomes sufficiently large, we say that the function has a **limit at negative infinity** and write

$$\lim_{x\to-\infty} f(x) = L.$$

From this definition, we define the horizontal asymptote.

Definition. [Horizontal Asymptotes] If $\lim_{x\to\infty} f(x) = L$ **or** $\lim_{x\to-\infty} f(x) = L$, we say that the line $y = L$ has a **horizontal asymptote** of f.

Example. [Functions with one horizontal asymptotes]

The function $f(x) = \frac{1}{x}$ has a horizontal asymptote at $y = 0$ since $\lim_{x\to\pm\infty} \frac{1}{x} = 0$.

The function $f(x) = \frac{x+1}{x}$ has a horizontal asymptote at $y = 1$ because

$$\lim_{x\to\pm\infty} \frac{x+1}{x} = \lim_{x\to\pm\infty} \left(1 + \frac{1}{x}\right) = 1.$$

Do notice that the horizontal asymptote is "global" in the sense that it is achieved at both infinities.

We highlight the qualifier "or" in the definition, with an example that demonstrate the possibility of having a horizontal asymptote at only one end (one infinity). This also means that checking for the limit of a function at one end and failing to find a limit is **insufficient** to conclude that it does not have a horizontal asymptote.

Example. The function $f(x) = \left(\frac{1}{2}\right)^x$ has a horizontal asymptote at $y = 0$ because

$$\lim_{x\to\infty} \left(\frac{1}{2}\right)^x = 0.$$

This is true because we are just taking larger and larger power of a fraction. We just check the other side,

$$\lim_{x\to-\infty} \left(\frac{1}{2}\right)^x = \infty.$$

Notice that

$$\lim_{x\to-\infty} \left(\frac{1}{2}\right)^x = \lim_{x\to-\infty} 2^{-x} = \lim_{y\to\infty} 2^y = \infty$$

with a change of variable $y = -x$, so $x \to -\infty$ corresponds to $y \to \infty$.

Example. [A function with two different horizontal asymptotes]

$f(x) = \tan^{-1}(x)$. This is a tricky one but you should think about why. You learned/reviewed in the Discussion about Inverse Functions. First of all, you should note that in order for this inverse to be well-defined, it is the inverse of $\tan(x)$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, or otherwise, the function $\tan(x)$ is not one-to-one (see After Class reading in Discussion 1). The inverse function $\tan^{-1}(x)$ has domain $(-\infty, \infty)$ and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (flipped from the original function $\tan(x)$).

Here, the limits we are interested in are

$$\lim_{x\to\infty} \tan^{-1}(x) \qquad \lim_{x\to-\infty} \tan^{-1}(x).$$

Now, suppose that these limits exist. So, we are confronted with the value $\tan^{-1}(\infty)$. To figure out its value, we write

$$L = \tan^{-1}(\infty) \implies \tan(L) = \infty.$$

So, one may ask, what angle L gives an infinite value, that is, what angle has a rise over run infinitely long? $L = \frac{\pi}{2}$.

Similarly,

$$L = \tan^{-1}(-\infty) \implies \tan(L) = -\infty \implies L = -\frac{\pi}{2}.$$

Thus, $\tan^{-1}(x)$ has **two** horizontal asymptotes at $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$.

Claim. A function cannot have more than two horizontal asymptotes. Why? The short answer is that such object will fail the vertical line test. The long answer requires the rigorous definition of a limit.

3 Infinite Limits at Infinity

Certainly, some functions will also "blow up" as $x \to \pm\infty$. One good example is in fact the set of all polynomials.

Definition. [Informal definition of Infinite Limits at Infinity]

We say a function f has an infinite limit at infinity and write

$$\lim_{x\to\infty} f(x) = \infty$$

if $f(x)$ becomes arbitrarily large for x sufficiently large.

We say a function f has a negative infinite limit at infinity and write

$$\lim_{x\to\infty} f(x) = -\infty$$

if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large for x sufficiently large. The limit as $x \to -\infty$ can be defined analogously.

Example.

$$\lim_{x\to\infty} 3x^4 = +\infty$$
$$\lim_{x\to-\infty} 3x^4 = +\infty$$
$$\lim_{x\to\infty} x^3 = +\infty$$
$$\lim_{x\to-\infty} x^3 = -\infty.$$

4 End Behaviour, Dominant Term and Oblique Asymptote

End Behaviours is the behaviour of a function as $x \to \pm\infty$. Only three things can happen at each of the function's ends.

1. Horizontal asymptotes, i.e., $f(x)$ approaches a flat line $y = L$.
2. The function blows up/down $f(x) \to \infty$ or $f(x) \to -\infty$.
3. The function does not approach a finite limit, nor does it approach ∞ or $-\infty$. In this case, the function may have some oscillatory behaviour (think $\sin(x)$ as $x \to \infty$).

4.1 Polynomials

For polynomials of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

we find

$$\begin{aligned}
\lim_{x\to\infty} p(x) &= \lim_{x\to\infty} \left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right) \\
&= \lim_{x\to\infty} a_n x^n \left(1 + \frac{a_{n-1}}{a_n}\frac{1}{x} + \cdots + \frac{a_1}{a_n}\frac{1}{x^{n-1}} + \frac{a_0}{a_n}\frac{1}{x^n}\right) \\
&\overset{\text{product rule}}{=} \left(\lim_{x\to\infty} a_n x^n\right) \lim_{x\to\infty} \left(1 + \frac{a_{n-1}}{a_n}\frac{1}{x} + \cdots + \frac{a_1}{a_n}\frac{1}{x^{n-1}} + \frac{a_0}{a_n}\frac{1}{x^n}\right) \\
&= \lim_{x\to\infty} a_n x^n
\end{aligned}$$

So, the limit at infinity (the **end behaviour**) for a polynomial solely depends on the sign of the leading order coefficient a_n. That is,

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n = \begin{cases} +\infty, & a_n > 0 \\ -\infty, & a_n < 0 \end{cases}$$

However, for the limit at negative infinity, the oddness/evenness of n plays a role.

$$\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} a_n x^n = \begin{cases} +\infty, & a_n > 0, & n \text{ even} \\ -\infty, & a_n > 0, & n \text{ odd} \\ -\infty, & a_n < 0, & n \text{ even} \\ +\infty, & a_n < 0, & n \text{ odd} \end{cases}$$

Exercise. Stare into your soul in the mirror, and reason yourself through the results above for the limit at negative infinity.

Example.

$$\lim_{x\to\infty} \left(5x^3 - 2x^2 + 4\right) = \lim_{x\to\infty} \left(5x^3\right) = +\infty,$$

because we said that all we care about is the leading order term. Then, observe that also,

$$\lim_{x\to-\infty} \left(5x^3 - 2x^2 + 4\right) = \lim_{x\to-\infty} \left(5x^3\right) = -\infty.$$

4.2 Rational Functions

So, the last result about limits at infinities for polynomials convinces us that only the leading order term matters. Indeed, when x becomes large, the trailing orders don't match up enough; imagine comparing 10^{10} and 10^5.

A natural question then arises: can we compare polynomials by taking a ratio of them? How about rational functions in general? In the following examples, you will see a general approach to address these issues.

Example. Determining the End Behaviour for Rational Functions:

Strategy: divide and multiply by the **largest** power in the denominator.

1. [Numerator and Denominator have equal degree.]

$$\lim_{x\to\infty} \frac{3x-1}{2x+5} = \lim_{x\to\infty} \frac{3-\frac{1}{x}}{2+\frac{5}{x}} = \frac{3}{2}.$$

   This shows that the function $f(x) = \frac{3x-1}{2x+5}$ has a horizontal asymptote at $y = \frac{3}{2}$, incidentally, the ratio of the leading coefficients in the numerator and denominator.

2. [Numerator Degree is larger than Denominator]

$$\lim_{x\to\infty} \frac{3x^2+2x}{4x^3-5x+7} = \lim_{x\to\infty} \frac{\frac{3}{x}+\frac{2}{x^2}}{4-\frac{5}{x^2}+\frac{7}{x^3}} = 0.$$

   This shows that the function $f(x) = \frac{3x^2+2x}{4x^3-5x+7}$ has a horizontal asymptote at $y = 0$.

3. [Numerator Degree is smaller than Denominator]

$$\lim_{x\to\infty} \frac{3x^2+4x}{x+2} = \lim_{x\to\infty} \frac{3x+4}{1+\frac{2}{x}} = +\infty.$$

Now, looking at the last example, we found that the function tends to infinity. However, the way that it tends to infinity seems to follow an invisible line (see Desmos/WolframAlpha sketches). What is the line?

We perform long division of the expression

$$\frac{3x^2+4x}{x+2} = 3x - 2 + \frac{4}{x+2}.$$

Now, if you don't like long division, you can use algebraic ideas.

$$\frac{3x^2+4x}{x+2} = \frac{\boxed{3x^2+6x} - 2x}{x+2} = 3x - \frac{2x}{x+2} = 3x - \frac{\boxed{2x+4} - 4}{x+2} = 3x - 2 + \frac{4}{x+2}$$

where every boxed term is written as a constant multiple of the denominator. It takes experience and a little thought to see what's going on here, if long division is not your cup of tea.

Now, back to the result of the long division,

$$\frac{3x^2+4x}{x+2} = 3x - 2 + \frac{4}{x+2}.$$

Notice that, if x becomes sufficiently large, the term $\frac{4}{x+2}$ is quite small, relatively to $3x-2$. Thus, we can write

$$\lim_{x\to\pm\infty} \left(f(x) - (3x-2)\right) = \lim_{x\to\pm\infty} \frac{4}{x+2} = 0.$$

This suggests that as $x \to \pm\infty$, $f(x)$ approaches the line $y = 3x-2$. The line is known as the **oblique asymptote** for f.

Remark. **Oblique asymptotes** only occur when the numerator has larger degree than the denominator.

4.3 Radical Functions mixed in with Rational Functions

We just learned about the limit at infinity of ratios of polynomials, and the easy tell is exactly the largest power difference between the numerator and the denominator. What if now we have some pesky square roots in the expression?

Example. Find the limits $x \to +\infty$ and $x \to -\infty$ for $f(x) = \frac{3x-2}{\sqrt{4x^2+5}}$ and describe the end behaviour of f.

Solution. The most important idea here is to recognize that when x is very large, the 5 under the square root barely matters. Therefore, the denomator is "practically" $\sqrt{4x^2} = 2x$, which is matching the largest power of the numerator. So, effectively, f is a rational function with equal largest power in the numerator and denominator. We expect a finite number when we take limit at infinity.

The strategy is still to divide top and bottom by the largest power in the denominator, namely, $\sqrt{x^2} = |x|$

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x-2}{\sqrt{4x^2+5}} &= \lim_{x\to\infty} \frac{\frac{1}{|x|}(3x-2)}{\frac{1}{|x|}\sqrt{4x^2+5}} \\
&= \lim_{x\to\infty} \frac{\frac{3x}{|x|} - \frac{2}{|x|}}{\sqrt{\frac{4x^2+5}{x^2}}} \\
&\overset{x>0 \implies \frac{x}{|x|}=1, \quad |x|=x}{=} \lim_{x\to\infty} \frac{3-\frac{2}{x}}{\sqrt{4+\frac{5}{x^2}}} \\
&= \frac{3-0}{\sqrt{4+0}} \\
&= \frac{3}{2}.
\end{aligned}$$

For $x \to -\infty$, we find that

$$\begin{aligned}
\lim_{x\to-\infty} \frac{3x-2}{\sqrt{4x^2+5}} &= \lim_{x\to-\infty} \frac{\frac{1}{|x|}(3x-2)}{\frac{1}{|x|}\sqrt{4x^2+5}} \\
&= \lim_{x\to-\infty} \frac{\frac{3x}{|x|} - \frac{2}{|x|}}{\sqrt{\frac{4x^2+5}{x^2}}} \\
&\overset{x<0 \implies \frac{x}{|x|}=-1, \quad |x|=-x}{=} \lim_{x\to-\infty} \frac{-3+\frac{2}{x}}{\sqrt{4+\frac{5}{x^2}}} \\
&= \frac{-3-0}{\sqrt{4+0}} \\
&= -\frac{3}{2}.
\end{aligned}$$

This function then has two horizontal asymptotes at $y = \frac{3}{2}$ and $y = -\frac{3}{2}$.

Remark. **[Important Remark on Rational Functions with Even Power of Roots – Should Help HW1 E7.3(a)]**

Take the example above. The short note is that we should always divide top and bottom by $\frac{1}{|x|}$ instead of $\frac{1}{x}$ when there is a square root expression, regardless of whether it's in the numerator or denominator. This avoid the following possible complication.

Suppose if you just divide top and bottom by $\frac{1}{x}$, and let's just investigate the limit as $x \to -\infty$, we have

$$\lim_{x\to-\infty} \frac{3x-2}{\sqrt{4x^2+5}} = \lim_{x\to-\infty} \frac{\frac{1}{x}(3x-2)}{\boxed{\frac{1}{x}\sqrt{4x^2+5}}}.$$

Notice that the boxed denominator is a negative number since x is going down to negative infinity. Now, carrying on by factoring the $\frac{1}{x}$ into the square root, we find that

$$\frac{1}{x}\sqrt{4x^2+5} = \sqrt{\frac{4x^2+5}{x^2}} \quad \text{(This is wrong – see why below)}$$

This is actually an abusrd statement because the left hand side is a negative number, while the right hand side is always positive. In fact, one should put a negative sign to fully correct this issue, that is,

$$\frac{1}{x}\sqrt{4x^2+5} = -\sqrt{\frac{4x^2+5}{x^2}} \quad \text{(This is now correct as signs do align from both side)}\,.$$

But all of this messing around can be avoided by using $\frac{1}{|x|}$ instead, and discuss carefully what $|x|$ is, when x is positive or negative.

4.4 End Behaviour for a Variant

Consider the transcendental function

$$f(x) = \frac{2+3e^x}{7-5e^x}.$$

To study the end behaviour of this function as $x \to \infty$, we again divide top and bottom by the dominating term in the denominator, namely, e^x.

$$\begin{aligned}
\lim_{x\to\infty} \frac{2+3e^x}{7-5e^x} &= \lim_{x\to\infty} \frac{\frac{2+3e^x}{e^x}}{\frac{7-5e^x}{e^x}} \\
&= \lim_{x\to\infty} \frac{2e^{-x}+3}{7e^{-x}-5} \\
&= \frac{0+3}{0-5} \\
&= -\frac{3}{5}.
\end{aligned}$$

Now, as $x \to -\infty$, the exponential terms are being dominated by the constants. In fact, we find that

$$e^x \to 0, \quad x \to -\infty.$$

Therefore

$$\lim_{x\to-\infty} \frac{2+3e^x}{7-5e^x} = \frac{2+0}{7-0} = \frac{2}{7}.$$

The function has two horizontal asymptotes,

$$y = -\frac{3}{5}, \quad y = \frac{2}{7}.$$

4.5 Dominant Term

Earlier, we have shown that the end behaviour of a polynomial is completely characterized by the leading order term. In rational functions, we may also observe such "leading order" behaviour. This section combines the knowledge above and declares the **dominant term** in a function expression.

Example. Consider $f(x) = \frac{x^2-3}{2x-4}$. By long division, we find that

$$f(x) = \left(\frac{x}{2}+1\right) + \frac{1}{2x-4}.$$

For $|x|$ large, we deduce that

$$f(x) \approx \frac{x}{2}+1,$$

since $\frac{1}{2x-4}$ is near 0.

For $x \approx 2$, the term $\frac{1}{2x-4}$ is large in absolute value, so

$$f(x) \approx \frac{1}{2x-4}.$$

Therefore, it is useful to determine how the function behaves near points of controversy (here at ∞ and $x = 2$).

We say that $\frac{x}{2}+1$ dominates or is the **dominant term** when $x \to \infty$ or $x \to -\infty$, and we say that $\frac{1}{2x-4}$ dominates or is the **dominant term** when $x \to 2$.

Then, when one asks for the **dominant term** of a function without further instruction, you should ask, "as x tends to what?", because the answer depends on it.

Example. [Rational function with Equal Powers]

The function

$$f(x) = \frac{4x^2+5}{x^2-3x}$$

can be further written as

$$f(x) = 4 + \frac{12x+5}{x^2-3x}$$

So, as $x \to \infty$, $f(x) \to 4$, showing that "4" is the dominant term (being the horizontal asymptote).

Now, as $x \to 0$ or $x \to 3$, we see that $f(x)$ reaches its two vertical asymptotes, respectively. There, 4 is insignificant compared to the fraction $\frac{12x+5}{x^2-3x}$, which is very large. The latter is then the dominant term near $x = 0$ and $x = 3$, respectively.

Remark. (Optional) In truth, one should perform a partial fraction decomposition to the latter term

$$\frac{12x+5}{x^2-3x} = \frac{A}{x} + \frac{B}{x-3} = \frac{A(x-3)+Bx}{x(x-3)}$$

which suggests that

$$\begin{aligned} A + B &= 12 \\ -3A &= 5 \end{aligned}$$

This system has solution

$$A = -\frac{5}{3}, \quad B = \frac{41}{3}.$$

Thus,

$$\frac{12x+5}{x^2-3x} = -\frac{5/3}{x} + \frac{41/3}{x-3}$$

which suggests that as $x \to 0$, the dominant term is $-\frac{5/3}{x}$ and as $x \to 3$, the dominant term is $\frac{41/3}{x-3}$.

Example. [Dominant Term for non-rational Functions]

Let $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ and $g(x) = 3x^4$.

Looking at these two polynomials, we know that for $x \to \infty$ or $x \to -\infty$, $f(x)$ and $g(x)$ are quite similar, because they share the same dominant term $3x^4$. One can verify that

$$\begin{aligned} \lim_{x\to\pm\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\pm\infty} \frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{3x^4} \\ &= \lim_{x\to\pm\infty} \frac{\frac{1}{x^4}\left(3x^4 - 2x^3 + 3x^2 - 5x + 6\right)}{\frac{1}{x^4}3x^4} \\ &= \lim_{x\to\pm\infty} \frac{3 - \frac{2}{x} + \frac{3}{x^2} - \frac{5}{x^3} + 6}{3} \\ &= \frac{3}{3} \\ &= 1. \end{aligned}$$

This suggests that when $|x|$ is large, f and g are nearly identical.

However, as $x \to 0$, all the powers become very small, except for that lone "6" in $f(x)$, which stays put regardless of where x goes. Therefore, the dominanting term for $f(x)$ as $x \to 0$ is the constant 6, while $g(x) \to 0$ as $x \to 0$. The behaviour of the two functions as $x \to 0$ is then quite different.

Example. [Exponential vs reciprocal]

Consider $h(x) = e^x + \frac{1}{x^2}$. As $x \to \infty$, the reciprocal has insignificant contributions, but the exponential is quite large. So, the dominant term of $h(x)$ as $x \to +\infty$ is e^x.

However, as $x \to 0$, $e^x \to 1$ but the reciprocal grows without bounds. Thus, the dominant term of $h(x)$ as $x \to 0$ is $\frac{1}{x^2}$.

Lastly, as $x \to -\infty$, both terms become small, so there is no dominant term, as $h(x) \to 0$.

Example. How about $f(x) = \frac{e^x}{x^2}$? This belongs to Calculus II. Stay tuned.

MATH 1021 Lecture V
Continuity and Types of Discontinuities

August 29, 2024

After class reading:

1 Continuity

We have relied on "plugging in" $x = a$ as a method to evaluate a limit, which may seem to almost defeat the purpose of knowing what a limit is. But surely, it didn't defeat it. The limiting value does NOT require that we know the **function value** $f(a)$.

Example. Consider the piecewise function

$$f(x) = \begin{cases} 2, & x = 1, \\ 1, & x \neq 1. \end{cases}$$

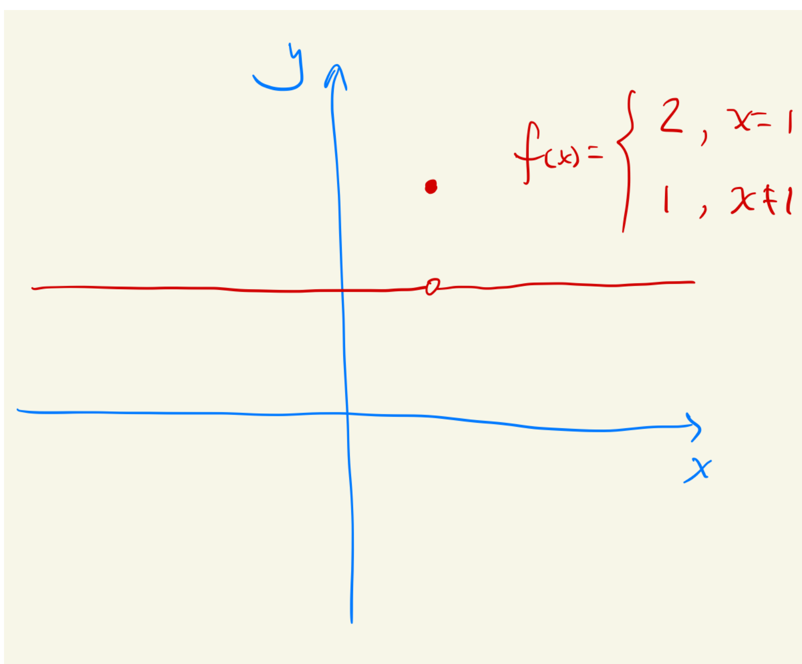


Figure 1.1:

We find that

$$\lim_{x \to 1^+} f(x) = 1 = \lim_{x \to 1^-} f(x)$$

because the function stays close to the limiting value 1 as $x \to 1$. However, the function value at $x = 1$ is NOT 1 but 2, as prescribed. The limit exists, but the function has a jump, or more mathematically, **discontinuous**.

In order to have "continuity" (before we define it), it seems that not only the limit needs to exist, but also the limiting value must be equal to the function value. Had the outlying point $(1, 2)$ been at $(1, 1)$, we would have perfect flat line which is obviously "continuous". Here is a proper mathematical definition of continuity using limits and function value (a perfect marriage between Calculus I and high school algebra).

Definition. [Continuity] A function $f(x)$ is **continuous at a point** $x = a$ if and only if the following three conditions are satisfied:

1. $f(a)$ is defined.
2. $\lim_{x\to a} f(x)$ exists.
3. $\lim_{x\to a} f(x) = f(a)$.

A function is **discontinuous at a point** $x = a$ if it fails to be continuous at a, i.e., one (or more) of the three conditions fail.

Remark. This definition lists the things you should check for continuity or discontinuity **in sequence**. If $f(a)$ is not even defined, then no need to check the next two – $f(x)$ is discontinuous at $x = a$. If $f(a)$ is defined, but the limit doesn't exist, then no need to go for the last criterion but conclude with discontinuity as well. Make sure you keep in mind an example to which each criterion corresponds (drawn in class).

Definition. We say that a function $f(x)$ is a **continuous function** when f is continuous at every point in its domain.

We say that a function $f(x)$ is a **continuous function** over the interval I when f is continuous at every $x \in I$.

So, what does this have to do with "plugging in" $x = a$ when evaluating a limit? Truthfully, when you "plugged in" $x = a$, you **assumed** that the function is continuous. Just look at criterion number 3 – isn't that just the very action of "plugging in"?

Example. The function $f(x) = \frac{x^2-4}{x-2}$ is discontinuous at $x = 2$ because the function value $f(2)$ is undefined. In fact, the function should be written more precisely as

$$f(x) = \begin{cases} x+2, & x \neq 2 \\ \text{undefined}, & x = 2 \end{cases}$$

Example. Check whether

$$f(x) = \begin{cases} -x^2+4, & x \leq 3 \\ 4x-8, & x > 3 \end{cases}$$

is continuous at $x = 3$. .

Solution. We need to check the three criteria.

1. $f(3) = -(3)^2 + 4 = -5$. So $f(3)$ is defined. The reason why we use $-x^2 + 4$ is because of the inclusive inequality $x \leq 3$ that define the function there.
2. $$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (4x-8) = 4(3) - 8 = 4;$$
   $$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} \left(-x^2+4\right) = -(3)^2 + 4 = -5,$$

   which shows that the limit does not exist at $x = 3$. We need look no further for criteria 3.

The function is NOT continuous at $x = 3$ (because the limit doesn't exist there).

Example. Determine if the function

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0 \end{cases}$$

is continuous at $x = 0$.

Solution. Again, we check the three criteria.

1. $f(0) = 1$ is defined.

2. $$\lim_{x\to 1^-} \frac{\sin x}{x} = 1 = \lim_{x\to 1^+} \frac{\sin x}{x}$$

   so the limit exists.

3. $$\lim_{x\to 1} \frac{\sin x}{x} = 1 = f(0),$$

   i.e., the limiting value is equal to the function value.

Altogether, the function is continuous at $x = 0$.

Theorem. *Polynomials, Rational and Trigonometric functions are continuous at every point* <u>*in their domains*</u>.

Example. For what values of x is the function $f(x) = \frac{x+1}{x-5}$ continuous?

Solution. Every x in the domain of f, i.e., all real numbers except $x = 5$.

Example. [Slightly more involved] State the interval(s) over which the function $f(x) = \frac{x-1}{x^2+2x}$ is continuous.

Solution. This is basically asking you to write the domain of the function in interval notations. We want to include all real numbers except $x = 0$ and $x = -2$. Therefore, the domain is

$$(-\infty, -2) \cup (-2, 0) \cup (0, \infty),$$

that is, a union of open intervals, without including the bad points. The function $f(x)$ is then continuous over **each** of the intervals here.

2 Types of Discontinuities

Continuity requires that all aforementioned three criteria are met. So, to be discontinuous, at least one criterion fails. We can then characterize the different types of discontinuity depending on which criterion fails.

Definition. [Types of Discontinuities] If $f(x)$ is discontinuous at $x = a$, then

1. f is a **removable discontinuity** at a if $\lim_{x\to a} f(x)$ exists, but is not equal to $f(a)$ or $f(a)$ is undefined.

   Example. $f(x) = \frac{x^2-4}{x-2}$ has a removable discontinuity because the limit does exist, but $f(a)$ is undefined.

2. f has a **jump discontinuity** at a if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist (excluding infinities), but are unequal.

   Example. $f(x) = \frac{|x|}{x}$ has a jump discontinuity at $x = 0$ because the left and right limits exist but do not coincide.

3. f has an **infinite discontinuity** at a if $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$.

   Example. Any function with a vertical asymptote at $x = a$. For example, $f(x) = \frac{1}{(x-a)^n}$ for n a positive integer.

MATH 1021 LECTURE VI
CONTINUOUS EXTENSIONS, COMPOSITE FUNCTION THEOREM AND THE INTERMEDIATE VALUE THEOREM

After class reading: Problem-Solving Strategy: Determining Continuity at a Point (page 181), Figure 2.37 (page 184), Examples 2.30-2.34 (page 185-187), Examples 2.37-2.38 (page 189-190).

1. Continuous Extensions

When a function is discontinuous at a point $x = a$ due to being undefined, we may define a function value so that the function *becomes* continuous. The procedure is called continuous extension.

Example. The function $f(x) = \frac{x^2-4}{x-2}$ is discontinuous at $x = 2$. In fact,

$$f(x) = \begin{cases} x+2, & x \neq 2 \\ \text{undefined}, & x = 2. \end{cases}$$

So, in order for f to be continuous, we need to give $f(2)$ a value. Certainly, it cannot be any value. Here, we find that the left and right limit of f as x approaches 2 yield the common limiting value 4. Therefore, setting $f(2) = 4$ makes f continuous over all real numbers.

In terms of its graph, the original function $f(x)$ is a line with a hole at $x = 2$. We simply filled in the hole by the act of setting $f(2) = 4$.

The only type of functions that you can extend continuously is the one with removable discontinuities. You just need to assign the function value such that it is equal to the limiting value at the point of discontinuity.

2. Composite Function Theorem

Function compositions help us evaluate more limits of functions that look complicated to begin with.

Theorem. *[Composite Function Theorem]*

If $f(x)$ is continuous at $x = L$ and $\lim_{x\to a} g(x) = L$, then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L).$$

Example. $\lim_{x\to\pi} \cos(x-\pi)$.

Using the fact that cosine is a continuous function, we write, using the Composite Function Theorem,

$$\lim_{x\to\pi} \cos(x-\pi) = \cos\left(\lim_{x\to\pi}(x-\pi)\right) = \cos(0) = 1.$$

Example. $\lim_{x\to 0} \sin\left(\frac{\pi}{2}\cos(x)\right)$.

A contrived example, but worth the exercise. Note that $\sin(x)$ is a continuous function (over all real numbers). Thus, by the Composite Function Theorem,

$$\begin{aligned} \lim_{x\to 0} \sin\left(\frac{\pi}{2}\cos(x)\right) &= \sin\left(\lim_{x\to 0} \frac{\pi}{2}\cos(x)\right) \\ &= \sin\left(\frac{\pi}{2}\lim_{x\to 0}\cos(x)\right) \\ &= \sin\left(\frac{\pi}{2}\cdot 1\right) \\ &= 1. \end{aligned}$$

3. The Intermediate Value Theorem

Continuous functions use one stroke and one stroke only to sketch its graph. Therefore, if it crosses the x-axis, the function values on the left and right of this crossing should differ by a sign, unless the x-axis is tangent to the function at the crossing. This observation is in fact a true mathematical statement, and is more generally concluded in the following Theorem.

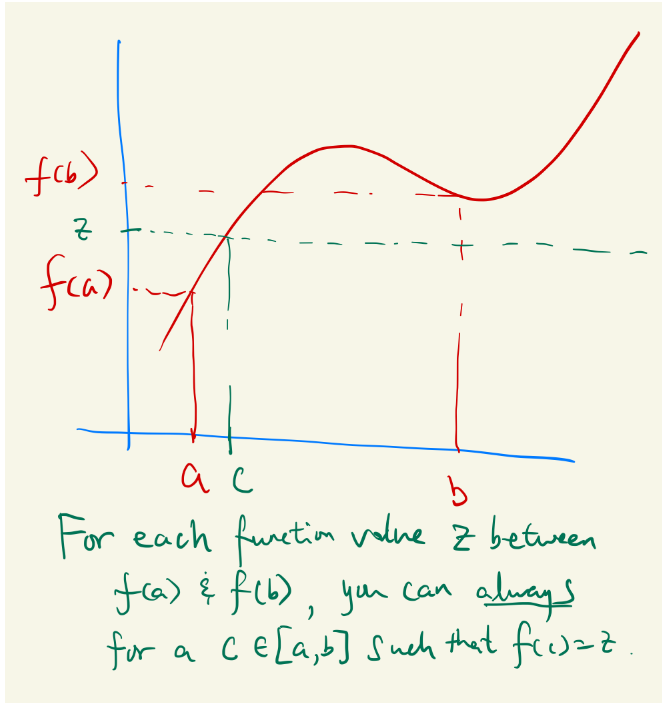


FIGURE 3.1.

Theorem. *[The Intermediate Value Theorem] Let f be continuous over a closed and bounded interval $[a, b]$. If z is any real number that lies between $f(a)$ and $f(b)$, then there is a number $c \in [a, b]$ such that $f(c) = z$.*

The intermediate value theorem is once again, **an existence theorem**. It tells that something must happen, without telling you where it precisely happens.

Corollary. *[Root-finding] Let f be continuous over a closed and bounded interval $[a, b]$. If $f(a)$ and $f(b)$ have different signs, then there exists a point $z \in [a, b]$ such that $f(z) = 0$.*

Proof. Suppose $f(a) < 0 < f(b)$. Then, take $z = 0 \in (f(a), f(b))$ in the Intermediate Value Theorem. Thus, there exists a number $c \in [a, b]$ such that $f(c) = 0$. □

Remark. A Corollary is an quick but meaningful consequence/special case of the Theorem it is attached to.

Example. Show that the function $f(x) = x - \cos x$ has at least one zero.

Solution. It would be great if you can solve it directly, namely, solve $x = \cos x$. Good luck!

The claim here is not to find where, but just state its existence, which is then aided by the Corollary of the IVT.

The job is to find two x-values that yield two function values of opposite signs. You always start easy.

$$f(0) = 0 - \cos(0) = -1 < 0,$$
$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \cos\left(\frac{\pi}{2}\right) = \frac{\pi}{2} > 0.$$

Therefore, there exists a point $c \in \left[0, \frac{\pi}{2}\right]$ such that $f(c) = 0$.

Can there be more? Of course.

4. The Bisection Method for Root-finding

The Intermediate Value Theorem helps us establish in which interval exists a root of a function, but it doesn't tell us **how** to find it. However, the essence of the IVT is that it is true as long as you are working with a continuous function on a closed and bounded interval. Therefore, we can apply the IVT multiple times until we are satisfied, detailed in the following algorithm.

(1) Find $a < b$ such that $f(a) < 0 < f(b)$ by trying a few values of a and b.
(2) Define $c = \frac{a+b}{2}$, the midpoint of $[a, b]$. Evaluate $f(c)$. IVT tells us the following.
 (a) If $f(c)$ has the same sign as $f(a)$, then we know there is a root on the new interval $[c, b]$.
 (b) If $f(c)$ has the same sign as $f(b)$, then we know there is a root on the new interval $[a, c]$.

(3) Repeat Step 2 on the new interval until $|f(c)|$ or c is sufficiently close to one of the new endpoints.

In essence, we are always halving the interval on either side, and we will eventually hone in on the root.

MATH 1021 LECTURE VII
RIGOROUS DEFINITION OF A LIMIT
LIMIT OF THE SLOPE OF A SECANT LINE

After class reading: Example 2.39 on page 195, Example 2.40 on page 198, Example 2.41 on page 199.

1. Rigorous Definition of a Limit

1.1. A Motivating Example and Definition.

Example 1. In HW1, we asked for a table of the function values for

$$f(x) = \frac{1-\cos x}{x},$$

for x-values very close to 0. We then made a very educated guess that judging from the table, the function tends to a limiting value of 0, i.e.,

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0.$$

Remark 2. We could just walk away with this answer, with no questions asked. But I can't help but ask, say, the smallest x we checked is

$$f\left(10^{-5}\right) = \frac{1-\cos\left(10^{-5}\right)}{10^{-5}} \approx 5.000 \times 10^{-6}.$$

Does this **guarantee** that the limiting value is 0? All we know is the function value at $x = 10^{-5}$, and **nothing else**. We just "guessed" that the function value is going down. But why down to, 0, specifically? No matter how small the x-value is, we have **zero guarantee** that the limiting value is what we expect. In other words, the limiting value remains an educated guess.

In the following definition, we define clearly what the symbol "$\to$" or the English word, "converges" means.

Definition 3. [The δ-ϵ definition of a limit of a function] For any $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ if $|x-a| < \delta$. We write

$$f(x) \to L, \quad \text{as } x \to a.$$

Remark. In plain English: no matter how small the number you tell me (ϵ), I can find a bunch of numbers close to a (within a radius of δ) so that the function values evaluated at these numbers is within ϵ reach of L.

Let's check the plain English definition and see how strong it is, on a simple example.

Example. Use the δ-ϵ definition of a limit of a function to **prove** that $\lim_{x\to 2} 5x = 10$.

Proof. In this case, $a = 2$ and $L = 10$. Remember, $L = 10$ is just an educated guess. You have NOT shown that the limiting value is actually 10.

Now, appealing to the definition. Say, you want the function value $f(x)$ to be within $\frac{1}{100}$ of 10. More precisely, you are saying that we want

$$|f(x) - 10| = |5x - 10| < \frac{1}{100}.$$

So, can I find δ so that $|x-2| < \delta$ guarantees $|5x-10| < \frac{1}{100}$? Noting that $|5x-10|$ is exactly $5|x-2|$, we simply choose

$$|x-2| < \frac{1}{5}\frac{1}{100} \implies |5x-10| < \frac{1}{100}.$$

Now, would you like to go smaller on the tolerance, say, you want

$$|f(x) - 10| < 10^{-3}, \quad 10^{-4}, \quad 10^{-5}, \ldots?$$

The relationship that $|5x-10| = 5|x-2|$ stays put, so we just set

$$\delta = \frac{1}{5}10^{-3}, \quad \frac{1}{5}10^{-4}, \quad \frac{1}{5}10^{-5}, \ldots.$$

In general, whatever ϵ you tell me, we choose $\delta = \frac{\epsilon}{5}$ so that the inequality

$$|f(x) - 10| < \epsilon$$

always holds. $\square$

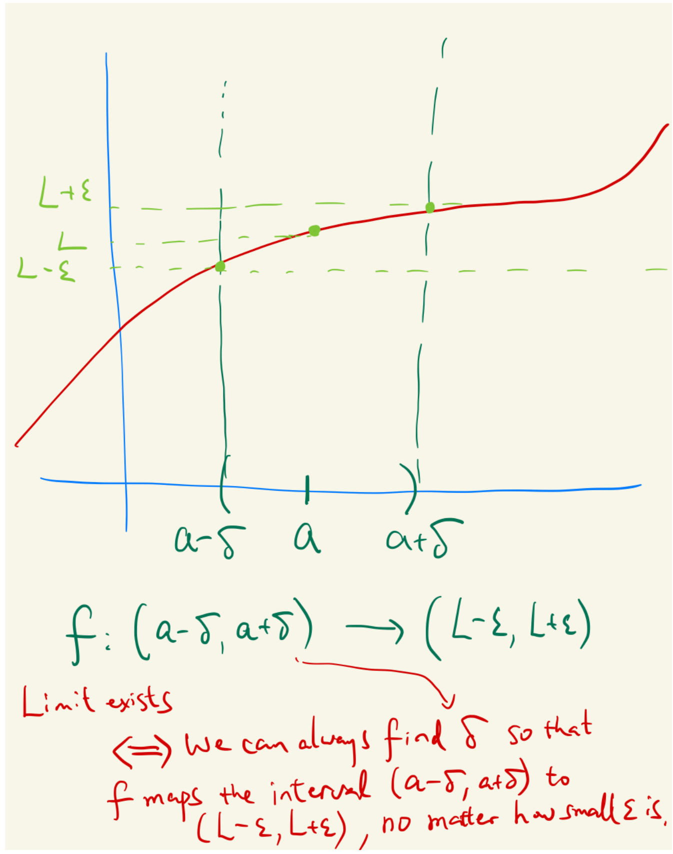


FIGURE 1.1.

1.2. **Strategy of Proving a Limit.** In proving that $\lim_{x\to a} f(x) = L$ (with L being the educated guess), we adopt the following steps:

(1) Begin with "Let $\epsilon > 0$", stating that we allow any tolerance, however small.
(2) Then, the punchline is to find δ (as possibly a function of ϵ). We say, "Choose $\delta =$".
(3) Choosing δ means you've drawn how far you are allowed to stray away from a, that is, you write

$$\text{Assume } 0 < |x-a| < \delta.$$

(4) Based on this assumption, you want to show

$$|f(x) - L| < \epsilon$$

knowing that you **must use** $0 < |x-a| < \delta$ at some point.
(5) Conclude with $\lim_{x\to a} f(x) = L$.

1.3. **More Examples.**

Example. Show that $\lim_{x\to -1}(4x+1) = -3$.

Proof. Let's follow the steps. □

(1) Let $\epsilon > 0$.
(2) Need to choose δ for $|x-(-1)| = |x+1| < \delta$.
(3) The statement we need to prove, under the assumption of step 2, is

$$|(4x+1)-(-3)| = |4x+4| = 4|x+1| \overset{(\mathbf{A})}{<} \epsilon.$$

(4) But we already know that $|x+1| < \delta$. It suffices to choose $\delta = \frac{\epsilon}{4}$ so that

$$4|x+1| < 4\delta = \epsilon.$$

We use step 2 at $(\mathbf{A})$.
(5) Conclude with $\lim_{x\to -1}(4x+1) = -3$.

2. Definition of a Derivative

In HW1, we have already encountered a useful concept that will motivate the notion of a derivative. Let's review it.

Example. Consider the parabola $y = x^2$. Find the slope of the secant line from $(1, 1)$ to $\left(\frac{5}{4}, \frac{25}{16}\right)$.

$$m_{\text{sec}} = \frac{\frac{25}{16} - 1}{\frac{5}{4} - 1} = \frac{\frac{9}{16}}{\frac{1}{4}} = \frac{9}{4}.$$

Definition 4. Let f be a function defined on an interval I containing a. If $x \neq a$ is in I, then

$$Q = \frac{f(x) - f(a)}{x - a}$$

is a **difference quotient**.

Definition. Also, if $h \neq 0$ is chosen so that $a + h$ is in I, then

$$Q = \frac{f(a + h) - f(a)}{h}$$

is a **difference quotient with increment** h.

Remark. Sometimes, I refer to h as "perturbation", "a little kick", "a small deviation in x", all to give you a mental image of what h is doing.

The HW problem then asks how you may improve the approximation of the slope of the tangent line at $(1, 1)$. We shall find a point on $f(x) = x^2$ a little closer to $(1, 1)$, and calculate the slope of the secant line there. If you eventually move the point "so close" to $(1, 1)$, we might get a more accurate approximation of the slope of the tangent line at $(1, 1)$. This act of moving the point closer to $(1, 1)$ can be understood as a limiting process, defined as follows.

Definition 5. Let $f(x)$ be a funciton defined in an open interval containing a. The **tangent line** to $f(x)$ at a is the line passing through the point $(a, f(a))$ having slope

$$m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Equivalently, we may define the slope of this tangent line as

$$m_{\tan} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

provided that this limit exists.

Example. Find the equation of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Solution. Two ingredients: find slope first, and then use point-slope form.

$$\begin{aligned} m_{\tan} &= \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} \\ &= \lim_{x \to 3} \frac{x^2 - 9}{x - 3} \\ &= \lim_{x \to 3} (x + 3) \\ &= 6. \end{aligned}$$

Since the point $(3, f(3)) = (3, 9)$ also lives on the line, we have, in point-slope form,

$$y - 9 = 6(x - 3).$$

Now, using the other formulation,

$$
\begin{aligned}
m_{\tan} &= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \\
&= \lim_{h\to 0} \frac{(3+h)^2-9}{h} \\
&= \lim_{h\to 0} \frac{h^3+6h+9-9}{h} \\
&= \lim_{h\to 0} \frac{h^3+6h}{h} \\
&= \lim_{h\to 0} \left(h^2+6\right) \\
&= 6.
\end{aligned}
$$

We found the same answer.

Mathematicians are lazy. We hate saying the word "slope of the tangent line" over and over again. Since this concept appears quite often, in kinematics, physics, finance, biology, population dynamics, chemical reaction, you name it, we give it a name – the derivative. Calculating the derivative of a function $f(x)$ at $x=a$ is **equivalent** to finding the slope of the tangent line of $f(x)$ at $x=a$. We typically write it as

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a},$$

or

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

3. Kinematic Revisited

A physical object traveling in space follows a position function $s(t)$ as a function of time t. The average velocity over a time interval $[b,t]$, is the different quotient

$$v_{\text{ave}} = \frac{s(t)-s(b)}{t-b},$$

also known as the slope of the secant line (of the position function). Thus, by applying a limit, that is, taking $t\to b$, we can find the **(instantaneous) velocity** at some time t,

$$v_{\text{insta}} = \lim_{t\to b} \frac{s(t)-s(b)}{t-b} = s'(t),$$

which is incidentally, the **derivative** of the position function $s(t)$.

If you are provided a graph of $s(t)$, then estimating the slope of the tangent line at a point of interest $t=t_0$ will give an estimate of the velocity.

MATH 1021 LECTURE VIII
DERIVATIVE AS A FUNCTION AND DIFFERENTIABILITY

Recall that we say a function is **continuous** when it is continuous at every point of its domain. What do we call a function that has a well-defined derivative at every point of its domain? A **differentiable** function. Last class, we know how to find the derivative **at a particular point**. The idea today is to find a general expression for the derivative of a given function, at **any** given point. This is equivalent to finding the **derivative** of a given function, **as a function**.

1. Derivative as a Function

Example. Find the derivative of $f(x) = x^2$ at $a = 1$, $a = 2$ and $a = 3$. Observe a pattern, and then determine the derivative $f'(x)$ as a function.

$$\begin{aligned} f'(1) &= \lim_{x\to 1} \frac{f(x) - f(1)}{x-a} = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 2 \\ f'(2) &= \lim_{x\to 1} \frac{f(x) - f(2)}{x-a} = \lim_{x\to 1} \frac{x^2-4}{x-2} = \lim_{x\to 2}(x+2) = 4 \\ f'(3) &= \lim_{x\to 1} \frac{f(x) - f(3)}{x-a} = \lim_{x\to 1} \frac{x^2-9}{x-3} = \lim_{x\to 3}(x+3) = 6. \end{aligned}$$

It looks like we have a pattern, and would think that given any point x, the derivative should be

$$f'(x) = 2x.$$

We prove this directly using the h-formulation.

$$\begin{aligned} f'(x) &\overset{\text{definition of a derivative}}{=} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &\overset{\text{definition of } f(x)}{=} \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &\overset{\text{expand the square}}{=} \lim_{h\to 0} \frac{\left(x^2+2xh+h^2\right) - x^2}{h} \\ &\overset{\text{cancel the } x^2}{=} \lim_{h\to 0} \frac{2xh+h^2}{h} \\ &\overset{\text{divide by } h}{=} \lim_{h\to 0} 2x+h \\ &= 2x. \end{aligned}$$

You can also prove this using the other formulation. Note that where the given point x shows up in this definition.

$$\begin{aligned} f'(x) &= \lim_{y\to x} \frac{f(y) - f(x)}{y-x} \\ &= \lim_{y\to x} \frac{y^2-x^2}{y-x} \\ &= \lim_{y\to x} \frac{(y-x)(y+x)}{y-x} \\ &= \lim_{y\to x}(y+x) \\ &= 2x. \end{aligned}$$

Example. Find the derivative of $f(x) = \sqrt{x}$.

Solution.

$$\begin{aligned}
f'(x) &\overset{\text{definition of a derivative}}{=} \lim_{y\to x}\frac{f(y)-f(x)}{y-x}\\
&\overset{\text{definition of } f(x)}{=} \lim_{y\to x}\frac{\sqrt{y}-\sqrt{x}}{y-x}\\
&\overset{\text{rationalization}}{=} \lim_{y\to x}\frac{\sqrt{y}-\sqrt{x}}{y-x}\frac{\sqrt{y}+\sqrt{x}}{\sqrt{y}+\sqrt{x}}\\
&= \lim_{y\to x}\frac{(y-x)}{(y-x)\sqrt{y}+\sqrt{x}}\\
&= \lim_{y\to x}\frac{1}{\sqrt{y}+\sqrt{x}}\\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}$$

Notice that the function $f'(x)$ has a vertical asymptote at $x=0$. This suggests that the function is NOT differentiable at $x=0$. From the sketch, you can see that the graph of $\sqrt{x}$ is vertical at $x=0$.

Of course, we can certainly check whether a function is differentiable or not at a given point, by studying the limit definition. A limit exists when its left and right limits coincide. Since a derivative is a limit, this "left" and "right" business translate directly to the existence of a derivative.

Definition. We say a function is **differentiable** at a point $x=a$, when both

$$f'_-(a)=\lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h},\ \text{the left derivative at } x=a$$
$$f'_+(a)=\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h},\ \text{the right derivative at } x=a$$

exist and are equal. In other words, f is differentiable at a point $x=a$ when its derivative exists at $x=a$.

Remark. Since these are derivatives, we can also write them in an alternate fashion.

$$f'_-(a)=\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a},\ \text{the left derivative at } x=a,$$
$$f'_+(a)=\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a},\ \text{the right derivative at } x=a.$$

The only change here is to approach a from the left or from the right.

Remark. Is this a lot to remember? No, it is not. All of this becomes simple when you can visualize what exactly is being done here. To internalize the definitions, one must verbalize – talk to your friends, your classmates, or directly to me. See the picture drawn in class.

Example. Is $f(x)=|x|$ differentiable everywhere in its natural domain?

Solution. We should write out the function in a piecewise fashion, due to the absolute value sign.

$$f(x)=\begin{cases} x, & x\geq 0,\\ -x, & x<0.\end{cases}$$

Away from $x=0$, the function is merely two lines. What's the slope of the tangent line of a line? Just the slope of the line. So,

$$f'(x)=\begin{cases} 1, & x>0,\\ -1, & x<0.\end{cases}$$

There is an issue near $x=0$. We check its left derivative using the alternate definition

$$f'_-(0)=\lim_{x\to a^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{|x|-0}{x-0}=\lim_{x\to 0^-}\frac{|x|}{x}=\lim_{x\to 0^-}\frac{-x}{x}=-1$$
$$f'_+(0)=\lim_{x\to a^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{|x|-0}{x-0}=\lim_{x\to 0^+}\frac{|x|}{x}=\lim_{x\to 0^+}\frac{x}{x}=1$$

We find that the left and the right limits differ, and therefore, the function is NOT differentiable at $x=0$.

In conclusion, the function $f(x)=|x|$ is differentiable in its natural domain except at $x=0$.

The derivative $f'(x)$ of a function $f(x)$ is also sometimes written as

$$\frac{d}{dx}(f(x)), \quad \frac{df}{dx}.$$

The "prefactor" $\frac{d}{dx}$ is not a multiplication, but an operator. We say that the act of taking a derivative (in the mathematical sense), is written as $\frac{d}{dx}$.

Why this prefactor, but not others? Recall that a derivatve is the limit of the slope of the secant lines. A slope is just "rise over run", i.e. for a function $y = f(x)$,

$$\frac{\Delta y}{\Delta x}.$$

We bring one point of the secant line to the other, we are shrinking both Δy and Δx at the same time, which then produces the derivative,

$$f'(x) = \frac{dy}{dx} = \frac{d}{dx}(y).$$

You could think of $\frac{dy}{dx}$ as "infinitesimal rise over run", or think of $\frac{d}{dx}$ as an "operator", here on the function $y = f(x)$.

2. Differentiability Implies Continuity, but Not Vice Versa

Theorem 1. *[Differentiability Implies Continuity] If f is differentiable at a point $x = c$ it its domain, then f is continuous at $x = c$.*

Solution. f being differentiable at $x = c$ means $f'(c)$ exists (is a number). We **want** to show, using the fact that $f'(c)$ exists, that $\lim_{x\to c} f(x) = f(c)$, the definition of continuity.

To prove an equality, we start with one side and try to arrive at the other.

$$\begin{aligned}
\lim_{x\to c} f(x) &\overset{\text{add and subtract the same term}}{=} \lim_{x\to c}(f(x) - f(c)) + f(c) \\
&\overset{f(c)\text{ is a constant}}{=} f(c) + \lim_{x\to c}\frac{f(x)-f(c)}{x-c}(x-c) \\
&\overset{\text{limit product rule}}{=} f(c) + \left(\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\right)\left(\lim_{x\to c}(x-c)\right) \\
&\overset{f'(c)\text{ exists by assumption}}{=} f(c) + f'(c)\cdot 0 \\
&= f(c).
\end{aligned}$$

The reason why we "add and subtract the same term" is because we want to "cook up" an expression that may make the Difference Quotient show up, so that we can make further use of the condition of the theorem.

Remark. The consequence of this theorem is that differentiable functions are continuous – anything you can do with continuous function, you can do it with differentiable ones.

Example. Continuity does NOT imply differentiability. The counterexample is

$$f(x) = |x|.$$

It's a continuous function, but not differentiable at $x = 0$.

MATH 1021 LECTURE IX
DIFFERENTIATION RULES AND APPLICATIONS

After class reading: Examples 3.29-3.32 in Section 3.3 of the OpenStax Textbook Volume I.

1. Recap and Motivation

In the previous lecture, we studied the derivative as a function,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

provided that this limit exists. We found simple expressions for the derivative of the parabola $f(x)=x^2$, i.e., $f'(x)=2x$. Now, what about a power function in general, $f(x)=x^n$ where n is an integer?

2. The Constant Rule

If $f(x)=c$ where c is a constant, then $f'(x)=0$. Why? HW3.

3. The Constant Multiple Rule

$$\frac{d}{dx}(kf(x))=k\frac{d}{dx}(f(x)),$$

i.e., you can **always** factor out the constant multiplier.

Example. Find $f'(x)$ if $f(x)=3x^2$.

Solution.

$$\frac{d}{dx}\left(3x^2\right)=3\frac{d}{dx}\left(x^2\right)=3(2x)=6x.$$

4. Sum and Difference

$$\frac{d}{dx}(f(x)+g(x))=\frac{d}{dx}(f(x))+\frac{d}{dx}(g(x))$$

or simply

$$[f(x)+g(x)]'=f'(x)+g'(x).$$

Example. Suppose $f(x)=2x^2+1$. Then,

$$f'(x)=2\cdot 2x+0=4x.$$

5. The Power Rule

In HW2, you have demonstrated that if $f(x)=x^3$, then

$$f'(x)=\lim_{z\to x}\frac{f(z)-f(x)}{z-x}=\lim_{z\to x}\frac{z^3-x^3}{z-x}=\lim_{z\to x}\frac{(z-x)\left(z^2+zx+x^2\right)}{z-x}=\lim_{z\to x}\left(z^2+zx+x^2\right)=3x^2.$$

Recalling that $f'(x)=2x$ if $f(x)=2x$, we seem to see a pattern.

$f(x)$	$f'(x)$
$x=x^1$	1
x^2	$2x$
x^3	$3x^2$
.	.
.	.
.	.

Somehow the power falls down first as a multiplier, and then the power is reduced by 1.

Theorem. *If $f(x)=x^n$, for n a positive integer, then $f'(x)=nx^{n-1}$.*

Proof. We follow the definition.

$$\begin{aligned}
f'(x) &= \lim_{z\to x} \frac{f(z)-f(x)}{z-x} \\
&= \lim_{z\to x} \frac{z^n - x^n}{z-x} \\
&= \lim_{z\to x} \frac{(z-x)\left(z^{n-1}+z^{n-2}x+z^{n-3}x^2+\cdots+zx^{n-2}+x^{n-1}\right)}{z-x} \\
&= \lim_{z\to x} \left(z^{n-1}+z^{n-2}x+z^{n-3}x^2+\cdots+zx^{n-2}+x^{n-1}\right) \\
&= x^{n-1}+x^{n-2}x+x^{n-3}x^2+\cdots+x\cdot x^{n-2}+x^{n-1} \\
&= \underbrace{x^{n-1}+x^{n-1}+x^{n-1}+\cdots+x^{n-1}+x^{n-1}}_{n \text{ of these}} \\
&= nx^{n-1}
\end{aligned}$$

□

Remark. Note that as of now, we only know the above result for n a positive integer. The proofs for n negative, and n a real number are in fact more involved than the above, and shouldn't be taken for granted.

Example. Find $\frac{dy}{dx}$ if $y = 10x^7$.

Solution. We use the power rule directly.

$$\frac{dy}{dx} = \frac{d}{dx}\left(10x^7\right) = 10 \cdot \frac{d}{dx}\left(x^7\right) = 10 \cdot 7x^{7-1} = 70x^6.$$

6. The Product Rule

Theorem.

$$\frac{d}{dx}\left(f(x)\,g(x)\right) = \frac{d}{dx}\left(f(x)\right)g(x) + f(x)\frac{d}{dx}\left(g(x)\right)$$

or

$$[f(x)\,g(x)]' = f'(x)\,g(x) + f(x)\,g'(x).$$

Proof. Define $j(x) = f(x)\,g(x)$. Then,

$$\begin{aligned}
[f(x)\,g(x)]' &= j'(x) \\
&= \lim_{h\to 0} \frac{j(x+h)-j(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)\,g(x+h)-f(x)\,g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)\,g(x+h)-f(x)\,g(x+h)+f(x)\,g(x+h)-f(x)\,g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} g(x+h) + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} f(x) \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \lim_{h\to 0} g(x+h) + f(x) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= f'(x)\,g(x) + f(x)\,g'(x).
\end{aligned}$$

□

Example. Find the derivative of $h(t) = \left(t^2-2\right)\left(3t^3-5t\right)$.

Solution. One could multiply this expression out and perform the Power Rule. Nothing wrong with that. But that requires more calculations which lead to a higher chance of mistakes.

$$\begin{aligned} h'(t) &= \left[\frac{d}{dt}\left(t^2-2\right)\right]\left(3t^3-5t\right)+\left(t^2-2\right)\frac{d}{dt}\left(3t^3-5t\right) \\ &= (2t)\left(3t^3-5t\right)+\left(t^2-2\right)\left(9t^2-5\right) \\ &= 6t^4-10t^2+9t^4-18t^2-5t^2+10 \\ &= 15t^4-33t^2+10. \end{aligned}$$

7. The Quotient Rule

This rule is infamous because of its ugly expression.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{f'(x)\,g(x)-f(x)\,g'(x)}{[g(x)]^2}.$$

NEVER DO THE FOLLOWING

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{f'(x)}{g'(x)}.$$

We omit the proof here but will return to it when we learn the Chain Rule.

Remark. I am against memorizing this formula because a quotient $\frac{f(x)}{g(x)}$ is simply a product of $f(x)$ and $[g(x)]^{-1}$. We will learn the Chain Rule next class which will help us find an expression for $\frac{d}{dx}[g(x)]^{-1}$, the derivative of the reciprocal of $g(x)$ (not the inverse). The quotient rule is just an applied product rule, nothing more, and the product rule is FAR EASIER to remember (and derive).

Because of this reason, I refuse to give an example right now.

8. Extended Power Rule

Theorem. *If k is a negative integer, then*

$$\frac{d}{dx}\left(x^k\right)=kx^{k-1}.$$

Proof. If k is a negative integer, then we set $n=-k$ which is then positive. Then, by the quotient rule

$$\frac{d}{dx}\left(x^k\right)=\frac{d}{dx}\left(\frac{1}{x^n}\right)=\frac{\left[\frac{d}{dx}(1)\right]x^n-1\frac{d}{dx}\left(x^n\right)}{\left(x^n\right)^2}=-\frac{nx^{n-1}}{x^{2n}}=-nx^{n-1-2n}=-nx^{-n-1}.$$

Now, since $k=-n$, we have

$$\frac{d}{dx}\left(x^k\right)=-nx^{-n-1}=kx^{k-1}.$$

$\square$

Example. Find $\frac{d}{dx}\left(x^{-5}\right)$.

Solution. Bring the power down, then subtract the power by 1.

$$\frac{d}{dx}\left(x^{-5}\right)=-5x^{-5-1}=-5x^{-6}.$$

Example. Find $\frac{d}{dx}\left(\frac{7}{x^3}\right)$.

Solution. Just leave the 7 alone.

$$\frac{d}{dx}\left(\frac{7}{x^3}\right)=7\frac{d}{dx}\left(x^{-3}\right)=7\left(-3x^{-3-1}\right)=-21x^{-4}.$$

MATH 1021 LECTURE X
DERIVATIVES OF TRIGONOMETRIC FUNCTIONS, THE CHAIN RULE AND THE QUOTIENT RULE

After class reading: **bottom of page 172, below eq (2.17) up to end of Example 2.25** (if nothing else is mandatory, then this is), the Section 1 below (everytime you think you run out of time to learn a formula); Discussion 1 on Function Compositions and Inverse Functions.

1. One General Comment about Formulas

Memorizing a formula takes up the same amount of space as memorizing a technique because the technique typically has a single punchline. One formula solves a single group of problems, but a technique may apply to various groups of problems. The gain from knowing a technique is immense. Developing the skills to derive formulas (or scientific principles) helps you succeed in thinking deeply about any subject. Memorizing a formula doesn't.

There will be "a point of no return" in college in terms of the habit of knowing the "why"s, because you will be so pressured by coursework and job search. It's a good time to develop such habit, NOW. Trust me. Things only become easier if this habit is formed early on, and you get to conquer many complex ideas with ease because your basic understanding is so strong.

2. Derivatives of Trigonometric Functions

The use of sine and cosine is ubiquitous in natural sciences, data analysis, engineering, finance and economics. They are the basic building block of modeling oscillations. Understanding their derivatives is one way to learn more about their properties.

We proceed directly with the definition of a derivative.

$$\begin{aligned}\frac{d}{dx}(\sin(x)) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\ &= \lim_{h\to 0}\frac{\sin x\cos h+\sin h\cos x-\sin x}{h}\\ &= \lim_{h\to 0}\frac{\sin x\cos h+\sin h\cos x-\sin x}{h}\\ &= \sin(x)\lim_{h\to 0}\frac{\cos h-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin h}{h}.\end{aligned}$$

We are kind of stuck now. These two limits are something we haven't encountered before explicitly (you've seen the cosine one but only produced a table to "guess" its limit). I will refer you to the brilliant explanation in the OpenStax text. Now, we take the following fact for granted:

$$\lim_{h\to 0}\frac{\sin h}{h}=1,\quad \lim_{h\to 0}\frac{\cos h-1}{h}=0.$$

Using these two limits, we continue the previous investigation and find

$$\frac{d}{dx}(\sin(x))=\sin(x)\cancelto{0}{\lim_{h\to 0}\frac{\cos h-1}{h}}+\cos(x)\cancelto{1}{\lim_{h\to 0}\frac{\sin h}{h}}=\cos(x).$$

Now, we may check for cosine.

$$\begin{aligned}\frac{d}{dx}(\cos(x)) &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}\\ &= \lim_{h\to 0}\frac{\cos x\cos h-\sin h\sin x-\cos x}{h}\\ &= \cos(x)\cancelto{0}{\lim_{h\to 0}\frac{\cos h-1}{h}}-\sin(x)\cancelto{1}{\lim_{h\to 0}\frac{\sin h}{h}}\\ &= -\sin(x).\end{aligned}$$

Collecting these results, we have the following Theorem

Theorem. $\frac{d}{dx}(\sin(x)) = \cos(x)$, $\frac{d}{dx}(\cos(x)) = -\sin(x)$.

You should commit this to memory, and must remember the negative sign for the derivative of cosine.

Remark 1. Note that the derivative of sine and cosine seem to "wrap around" itself. Indeed, we find that

$$\begin{aligned}\frac{d}{dx}(\sin(x)) &= \cos(x)\\ \frac{d^2}{dx^2}(\sin(x)) &= \frac{d}{dx}(\cos(x)) = -\sin(x)\\ \frac{d^3}{dx^3}(\sin(x)) &= \frac{d}{dx}(-\sin(x)) = -\cos(x)\\ \frac{d^4}{dx^4}(\sin(x)) &= \frac{d}{dx}(-\cos(x)) = \sin(x),\end{aligned}$$

that is, if you take the fourth derivative of $\sin(x)$, you get the function back exactly. The same case applies to cosine. This fact, in a future calculus class, somehow links to the imagery unit, $i = \sqrt{-1}$. What the serious heck is going on?

Remark 2. Look out for another way of verifying the derivative for sine and cosine after we learn the Chain Rule below.

3. The Chain Rule

3.1. **Motivation.** A horse is carrying a carriage on a dirt path. The amount of (energy) E (in calories) expended by the horse depends on the (distance) m (in miles) the horse walks. Also, the distance s walked by the horse depends on (time) t (in hours). If the horse expends 40 calories per mile and the horse walks at a speed of 8 mph, at what rate is the horse expending energy (calories/hour)?

Intuitively, one would multiply these two rates to get the rate of energy expenditure. How is it related to the Chain Rule? Consider the energy per mile as $\frac{dE}{ds}$ and the distance per hour is $\frac{ds}{dt}$. We want $\frac{dE}{dt}$, the rate of energy expenditure per unit time, with units in calories per hour. Intuitively, it is just a simple multiplication of the two given quantities,

$$\frac{dE}{dt} = \frac{dE}{ds} \cdot \frac{ds}{dt} = 40 \cdot 8 = 320 \text{ calories per hour.}$$

Looking at the dimensions of this equation, it seems that everything checks out.

In more proper mathematical terms, the energy of the horse, at a given time t, is $E(s(t))$, a composite function. So, the equation above, in essence, can be expressed as

$$\frac{d}{dt}E(s(t)) = \frac{d}{ds}(E(s)) \cdot \frac{d}{dt}(s(t)) = E'(s(t)) \cdot s'(t).$$

This is the celebrated Chain Rule.

3.2. **The Chain Rule.**

Theorem. *[The Chain Rule] Let f and g be functions. For all x in the domain of g for which g is differentiable at x, and f is differentiable at $g(x)$, the derivative of the composite function*

$$h(x) = (f \circ g)(x) = f(g(x))$$

is given by

$$h'(x) = f'(g(x))\, g'(x).$$

Alternatively, if y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Remark. The crucial task at hand here is to be able **identify** the components of a composite function. This was done in Discussion 1 and HW0.

Example. Identify the outer and inner function of the following composite functions and then find their derivatives.

(1) $h(x) = (3x^2 + 1)^9$.

Solution. You are always welcome to expand the sixth power. Be my guest. Email me either when you are done or when you graduate, whichever comes first.

Instead, we view h as a composite function

$$h(x) = f(g(x))$$

where $g(x) = 3x^2 + 1$ and $f(x) = x^9$. Thus, the derivative can be computed directly via the Chain Rule,

$$\begin{aligned} h'(x) &= f'(g(x))\, g'(x) \\ &= \left(\frac{d}{dx}\left[x^9\right] |_{3x^2+1}\right)\left(\frac{d}{dx}\left[3x^2+1\right]\right) \\ &= \left(\left[9x^8\right] |_{3x^2+1}\right)(6x) \\ &= 9\left(3x^2+1\right)^8 (6x) \\ &= 54x\left(3x^2+1\right)^8. \end{aligned}$$

(2) $h(x) = \sin\left(x^3\right)$.

Solution. The derivative of this function has no other way but the Chain Rule.

$$h(x) = f(g(x))$$

where $g(x) = x^3$ and $f(x) = \sin(x)$. Then,

$$\begin{aligned} h'(x) &= f'(g(x))\, g'(x) \\ &= \left(\frac{d}{dx}\left[\sin(x)\right] |_{x^3}\right)\left(\frac{d}{dx}\left[x^3\right]\right) \\ &= \left(\left[\cos(x)\right] |_{3x^2+1}\right)\left(3x^2\right) \\ &= 3x^2 \cos\left(3x^2+1\right). \end{aligned}$$

4. The Quotient Rule

Usually, the Quotient Rule is studied along with the Product Rule. However, the ugly nature of the Quotient Rule formula is not very inspiring because it tells you **nothing** about how it is derived. Now, equipped with the Product Rule and the Chain Rule, we can do it, **anytime we want to**.

Lemma. *If $g(x)$ is differentiable, then*

$$\frac{d}{dx}\left[\frac{1}{g(x)}\right] = -\frac{g'(x)}{\left[g(x)\right]^2}.$$

Proof. We use the Extended Power Rule (learned in the Discussion), and the Chain Rule. Consider $\frac{1}{g(x)}$ as a composite function with components

$$f(x) = \frac{1}{x}, \quad g(x) = g(x) \implies \frac{1}{g(x)} = f(g(x)).$$

Thus,

$$\begin{aligned} \frac{d}{dx}\left[\frac{1}{g(x)}\right] &= \frac{d}{dx}\left[f(g(x))\right] \\ &\overset{\text{the Chain Rule}}{=} f'(g(x))\, g'(x) \\ &= \left[-\frac{1}{x^2}\right] |_{g(x)}\, g'(x) \\ &= -\frac{g'(x)}{\left[g(x)\right]^2}. \end{aligned}$$

□

Remark. A Lemma is an assistant to a larger theorem and typically involves a slightly more lengthy calculation almost worthy as an independent result.

Example. Find the derivative of $f(x) = \csc(x)$.

Solution. Note that $f(x) = \csc(x) = \frac{1}{\sin(x)}$. We use the Lemma directly by identifying that $g(x) = \sin(x)$.

$$f'(x) = -\frac{\frac{d}{dx}[\sin x]}{\sin^2 x} = -\frac{\cos(x)}{\sin^2(x)}.$$

Sometimes, folks simplify this to

$$f'(x) = -\frac{\cos(x)}{\sin^2(x)} = -\frac{\cos(x)}{\sin(x)}\frac{1}{\sin(x)} = -\cot(x)\csc(x).$$

Theorem. *[The Quotient Rule] Let $f(x)$ and $g(x)$ be functions. If $g(x) \neq 0$, then*

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Proof. We proceed with the Product Rule on $f(x)$ and $\frac{1}{g(x)}$.

$$\begin{aligned}\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= f'(x)\frac{1}{g(x)} + f(x)\frac{d}{dx}\left[\frac{1}{g(x)}\right]\\ &= \frac{f'(x)}{g(x)} + f(x)\left[-\frac{g'(x)}{[g(x)]^2}\right]\\ &= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2}\\ &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.\end{aligned}$$

□

Example. Find the derivative of the following functions.

(1) $g(z) = \frac{\sin z}{z+1}$.

Solution. The Quotient Rule it is.

$$g'(z) = \frac{\frac{d}{dz}[\sin z](z+1) - (\sin z)\frac{d}{dz}[z+1]}{(z+1)^2} = \frac{(z+1)\cos z - \sin z}{(z+1)^2}.$$

You may use the Product Rule, and write

$$\begin{aligned}g'(z) &= \frac{d}{dz}[\sin z]\frac{1}{z+1} + \sin(z)\frac{d}{dz}\left[\frac{1}{z+1}\right]\\ &= \frac{\cos z}{z+1} + \sin(z)\left(-\frac{1}{(z+1)^2}\right)\\ &\overset{\text{Lemma}}{=} \frac{\cos z}{z+1} - \frac{\sin(z)}{(z+1)^2}.\end{aligned}$$

They are the same answer (after you combine the fractions).

(2) $s(t) = \frac{t^2}{t^3+1}$.

Solution. The Quotient Rule still.

$$\begin{aligned}s'(t) &= \frac{\frac{d}{dt}[t^2](t^3+1) - (t^2)\frac{d}{dt}[t^3+1]}{(t^3+1)^2}\\ &= \frac{(2t)(t^3+1) - (t^2)(3t^2)}{(t^3+1)^2}\\ &= \frac{2t^4 + 2t - 3t^4}{(t^3+1)^2}\\ &= \frac{2t - t^4}{(t^3+1)^2}.\end{aligned}$$

MATH 1021 LECTURE XI
THE INVERSE FUNCTION THEOREM AND IMPLICIT DIFFERENTIATION

1. Motivation of Inverse Functions

Why are inverse functions useful? Have you ever encountered them yet you did not notice? In fact,when you are solving a quadratic equation, you used the notion of inverses already. Suppose we look for the zeros of the function

$$f(x) = x^2 + 3x + 1.$$

Factoring seems difficult, so we resort to the powerful quadratic formula,

$$x_{1,2} = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot 1}}{2}.$$

Note that your original equation is **quadratic**, and now the formula for the roots involves **square roots**. Of course, we know that taking a square root undoes a square (the highest power in a quadratic equation). We say that these two operations are **inverses** of each other. So, is it a coincidence here this inverse relationship somehow show up here?

Now, what if the question is a little harder: find the zeros of

$$g(x) = x^3 + 3x + 1.$$

You are specifically looking for the x^*'s such that $g(x^*) = 0$. The function inverse, written as $g^{-1}(y)$ (not the reciprocal $\frac{1}{g(x)} = [g(x)]^{-1}$), satisfies,

$$g^{-1}(g(x)) = x.$$

In particular, this identity works for the unknown zeros

$$g^{-1}(g(x^*)) = x^* \implies g^{-1}(0) = x^*.$$

This equation implies that if we know the form of $g^{-1}(y)$, all we need to do is to evaluate it at $y = 0$ to obtain the solutions x^*.

2. Derivative of an Inverse Function

In reality and in general, finding the exact inverse function is a hard problem. So, to learn any information about the inverse would be useful. We learned from earlier sections that the derivative of a function helps us trace out where the function is increasing/decreasing, which is quite a useful piece of information.

To visualize the inverse of a function $f(x)$, one can flip $f(x)$ across the line $y = x$. Suppose the tangent line at $(a, f(a))$, namely, has slope $f'(a) = \frac{p}{q}$. Reflecting this tangent line across $y = x$, one would expect that the roles of rise and run are reversed. This suggests that on the inverse function, the point $(f(a), a)$ would have slope $\left(f^{-1}\right)'(f(a)) = \frac{q}{p}$. Thus, we deduce the intuitive result

$$\left(f^{-1}\right)'(f(a)) = \frac{1}{f'(a)}.$$

Now, since f is invertible at $x = a$, there must be a value $b \in R(f)$ such that $f(a) = b \implies a = f^{-1}(b)$. Substituting $a = f^{-1}(b)$, and using the fact that $f\left(f^{-1}(b)\right) = b$, we have

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(b)\right)}.$$

Theorem. *[the Inverse Function Theorem]*

If f is differentiable on some interval I, and f' is never zero on I, then f^{-1} is differentiable everywhere on its domain (the range of f). Furthermore, given a point b in the domain of f^{-1}, we have

$$\left(f^{-1}\right)'(b) = \frac{1}{f'\left(f^{-1}(b)\right)}.$$

Alternatively, if $g(x)$ is the inverse of $f(x)$, then

$$g'(x) = \frac{1}{f'(g(x))}.$$

Proof. The first claim is not easy to prove, but it is motivated by the intuition we developed before the Theorem – in fact, the condition on f' is never zero must be assumed or otherwise the reciprocal in the equation is undefined.

We prove the second claim (the formula) via the Chain rule. We begin with the identity

$$f\left(f^{-1}(x)\right) = x$$

viewing the left-hand-side as a composite function. Write $g(x) = f^{-1}(x)$. Taking $\frac{d}{dx}$ on both sides, we have

$$\begin{aligned} \frac{d}{dx}\left(f\left(g(x)\right)\right) &= 1 \\ \implies f'\left(g(x)\right) g'(x) &= 1 \\ \implies g'(x) &= \frac{1}{f'\left(g(x)\right)} \end{aligned}$$

or

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)},$$

should $f'\left(g(x)\right) \neq 0$. □

Example. Use the inverse function theorem to find the derivative of $g(x) = \frac{x+2}{x}$. Compare the resulting derivative to that obtained by differentiating the function directly.

Solution. The inverse of $g(x)$ is $f(x) = \frac{2}{x-1}$. How did we get this? Write $y = \frac{x+2}{x}$, and then swap x and y, and solve for y,

$$x = \frac{y+2}{y} \implies xy = y + 2 \implies xy - y = 2 \implies y(x-1) = 2 \implies y = \frac{2}{x-1}, \quad x \neq 1.$$

By the Inverse Function Theorem,

$$g'(x) = \frac{1}{f'\left(g(x)\right)}.$$

So we need $f'(x) = -\frac{2}{(x-1)^2}$ via Power Chain Rule [see Useful Formulas and their Proofs on Canvas]. Therefore,

$$g'(x) = \frac{1}{f'\left(g(x)\right)} = -\frac{1}{\frac{2}{(g(x)-1)^2}} = -\frac{\left(g(x)-1\right)^2}{2} = -\frac{\left(\frac{x+2}{x}-1\right)^2}{2} = -\frac{\left(\frac{2}{x}\right)^2}{2} = -\frac{2}{x^2}.$$

Computing the derivative directly, we find

$$\frac{d}{dx}\left(g(x)\right) = \frac{d}{dx}\left(\frac{x+2}{x}\right) = \frac{d}{dx}\left(1 + \frac{2}{x}\right) = -\frac{2}{x^2},$$

as expected.

Example. Use the Inverse Function Theorem to find the derivative of $g(x) = x^{\frac{1}{3}}$.

Solution. The inverse of cube root is the cubic function $g^{-1}(x) = f(x) = x^3$. We also know that $f'(x) = 3x^2$. Thus, by the Inverse Function Theorem,

$$g'(x) = \frac{1}{f'\left(g(x)\right)} = \frac{1}{3\left[g(x)\right]^2} = \frac{1}{3x^{2/3}}.$$

Theorem. *[Extended Power Rule for Rational Exponents]*

If n is a positive integer, and m any integer, then

$$\frac{d}{dx}\left(x^{m/n}\right) = \frac{m}{n} x^{\frac{m}{n}-1}.$$

Proof. **READ IT** on Page 301 of the OpenStax textbook. □

Example. Find the derivative of $g(t) = (t + \sin t)^{\frac{1}{3}}$.

Solution. We apply the Power Chain Rule, with the Power part extended by the Theorem above. Recognize that

$$g(t) = f\left(h(t)\right)$$

where $h(t) = t + \sin t$ and $f(t) = t^{\frac{1}{3}}$. Thus,

$$\begin{aligned} \frac{d}{dt}[g(t)] &= f'(h(t))\,h'(t) \\ &= \frac{1}{3}[h(t)]^{\frac{1}{3}-1}(1+\cos t) \\ &= \frac{1}{3}(t+\sin t)^{-\frac{2}{3}}(1+\cos t). \end{aligned}$$

3. Special Case 1: Derivative of Inverse Trigonometric Functions

This section is devoted to finding the derivatives of inverse trigonometric functions.

Theorem.

$$\begin{aligned} \frac{d}{dx}\sin^{-1}(x) &= \frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx}\cos^{-1}(x) &= \frac{-1}{\sqrt{1-x^2}} \\ \frac{d}{dx}\tan^{-1}(x) &= \frac{1}{1+x^2} \\ \frac{d}{dx}\cot^{-1}(x) &= \frac{-1}{1+x^2} \\ \frac{d}{dx}\sec^{-1}(x) &= \frac{1}{|x|\sqrt{x^2-1}} \\ \frac{d}{dx}\csc^{-1}(x) &= \frac{-1}{|x|\sqrt{x^2-1}} \end{aligned}$$

Proof. We prove the first one about inverse sine. You do the rest in the Extra Credit Assignment. You are HIGHLY RECOMMENDED to do it.

The inverse of $\sin^{-1}(x)$ is the function $f(x) = \sin(x)$ (duh!). So,

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{f'\left(\sin^{-1}(x)\right)} = \frac{1}{\cos\left(\sin^{-1}(x)\right)}.$$

To figure out what $\cos\left(\sin^{-1}(x)\right)$ is, consider a right triangle, with an angle $\theta = \sin^{-1}(x)$. This means $\sin\theta = x$, which is the opposite divided by hypotenuse. So, this triangle now has an angle θ, with opposite of length x and a hypotenuse of length 1. What would be $\cos\left(\sin^{-1}(x)\right) = \cos\theta$? Just adjacent over hypotenuse,

$$\cos\left(\sin^{-1}(x)\right) = \cos\theta = \frac{1}{\sqrt{1-x^2}}$$

where $\sqrt{1-x^2}$ is the length of the adjacent.

More simply put, DRAW THE TRIANGLE and figure it out.

For the extra credit, perform a similar procedure. $\square$

MATH 1021 LECTURE XIV
EXPONENTIAL AND LOGARITHMIC FUNCTIONS AND THEIR DERIVATIVES

1. Construction of the Transcendental Number e

Suppose you deposit one dollar into a bank that pays you 100% interest at the end of year 1. Sounds like a good business as you double your principal every year. So, you decide to retrieve both your interest and your principal, totaling 2 dollars.

Knowing your time horizon is one year, the bank pushes out a new plan that allows you to compound your interest semi-annually. In other words, your interest earned over sixth months will be reinvested, but the semi-annual interest is now cut to 50%. Take again your one dollar for example. Now, after sixth months, you have 1 dollar of principal, and 0.5 dollar of interest, totalling 1.5 dollars in the account. This amount is now reinvested for the second half of the year, and by the end of the year, you will have 1.5 dollars of new principal, and 0.75 dollars of interests, totaling 2.25 dollars. Hmm, this is more than 2 dollars than Plan A. Will the bank then allow you to compound every quarter (with a quarterly interest of 25%), every month (with a monthly interest of 12.5%), every day, or then every hour?!

Is there a general formula for how often you compound your interest and the total amount of deposit? Consider the semi-annual example. After sixth months, we have $P = \left(1+\frac{1}{2}\right)$ dollars, and then we reinvest it to total $P\left(1+\frac{1}{2}\right) = \left(1+\frac{1}{2}\right)^2$ at the end of the year. Similarly, if you do this quarterly, you will have $\left(1+\frac{1}{4}\right)^4$. In general

How often do you compound in a year?	Total Deposit
1 (annually)	$(1+1)=2$
2 (semi-annually)	$\left(1+\frac{1}{2}\right)^2 = 2.25$
4 (quarterly)	$\left(1+\frac{1}{4}\right)^4 = 2.44140625$
12 (monthly)	$\left(1+\frac{1}{12}\right)^{12} = 2.56578451395$
.	.
.	.
n times	$\left(1+\frac{1}{n}\right)^n$

Do you see a limiting pattern now? Well, at least the money is growing, so it makes sense that we compound more often. If technology allows, you should compound infinitely often, namely, have a total amount of deposit as

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.718281828459045\ldots$$

where the symbol := means "defined as".

In mathematics, when we want to show that a sequence converges, we refer to the monotone convergence theorem, i.e., a bounded monotone (purely increasing or decreasing) sequence always converges. The sequence $\left(1+\frac{1}{n}\right)^n$ as you observed is always increasing, but is bounded below by 1 (duh!) and bounded above by 3 (not easy to see). So the sequence has a limit. **We call this limiting value e, the Euler number (though this number was discovered by Jacob Bernoulli in 1685).**

2. The Exponential Function

Now, we learn about the function $f(x) = e^x$. Since e is a number greater than 1, we have its end behaviour as

$$\lim_{x\to\infty} e^x = \infty;$$
$$\lim_{x\to-\infty} e^x = 0.$$

Noting further that $f(0) = e^0 = 1$, we now have good idea of what it is. Here is a collection of the exponential function's properties.

Proposition. *Given $f(x) = e^x$, then we have*

(1)

$$Dom(f) = \mathbb{R} = (-\infty, \infty).$$

(2)
$$R(f) = \mathbb{R}^{+} = (0, \infty).$$
(3)
$$\lim_{x\to\infty} e^{x} = \infty.$$
(4)
$$\lim_{x\to-\infty} e^{x} = 0.$$
(5)
$$e^{x+y} = e^{x}e^{y}.$$
(6) *e^x is one-to-one (see Discussion 1).*

3. The Natural Logarithm

The logarithm comes from a simple question. Suppose $2^x = 5$, i.e., to what power of 2 is 5? This should be a number between 2 and 3, but is still hard to quantify. As a responsible mathematician, we **define** that such number is
$$x = \log_2 5,$$
read as log base 2 of 5. Here, you should always associate this expression with the equation $2^x = 5$.

Now, of course, $\log_2 2 = 1$, $\log_2 = 1 = 0$. In fact, we make the base general, and define the function
$$y = f(x) = \log_a(x).$$
We must have $a > 0$ but $a \neq 1$ (because $\log_1(x)$ is undefined for $x \neq 1$, but multi-valued for $x = 1$).

Proposition. *Given $y = f(x) = \log_a(x)$, we have*

(1)
$$Dom(f) = (0, \infty).$$
(2)
$$R(f) = \mathbb{R} = (-\infty, \infty).$$
(3)
$$\log_a(1) = 0.$$
(4)
$$\log_a(a) = 1.$$
(5)
$$\lim_{x\to\infty} \log_a(x) = \infty, \quad a > 1.$$
(6)
$$\lim_{x\to0^+} \log_a(x) = -\infty.$$
(7)
$$\log_a(b^x) = x\log_a(b)$$
(8)
$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y).$$
$$\log_a(xy) = \log_a(x) + \log_a(y)$$
(9) *$\log_a(x)$ is one-to-one (see Discussion 1).*

Remark 1. Few remarks about each part.

(1) Note that the domain and range of the logarithm function is the exactly the opposite of those of the exponential function. We show later that a^x and $\log_a(x)$ are inverses of each other.
(2) Why $\log_a(1) = 0$? Well, a to the what power gives 1? a^0.
(3) Why $\log_a(a) = 1$? Well, a to the 1 power gives a.
(4) Why $\lim_{x\to\infty} \log_a(x) = \infty$ for $a > 1$? Well, if $x \to \infty$, then a needs to be raised to an infinite power to achieve infinity.
(5) Why $\lim_{x\to0^+} \log_a(x) = -\infty$ for $a > 1$? $a^{-\infty} = \left(\frac{1}{a}\right)^{\infty}$ but since $a > 1$, then $0 < \frac{1}{a} < 1$. Raising a fraction to a large power shrinks to zero, from the positives (hence 0^+).
(6) Why $\log_a b^x = x\log_a b$, namely, the logarithm extracts exponents? Let
$$y = \log_a b^x \implies a^y = b^x \implies a^{\frac{y}{x}} = b \implies \log_a b = \frac{y}{x} \implies y = x\log_a b.$$

Theorem. *If $f(x) = \log_a(x)$ and $g(x) = a^x$, then $f(x) = g^{-1}(x)$, i.e., they are inverses of each other.*

Proof. Simpler than you think. It's all by definitions. We want to show that $f(g(x)) = x$.

$$f(g(x)) = \log_a(a^x).$$

This is a curious expression, whose value is understood as: to what power of a is a^x? Well, x. Thus,

$$f(g(x)) = \log_a(a^x) = x.$$

You can do this the other way.

$$g(f(x)) = a^{\log_a x}$$

This is another curious expression, which we set

$$y = a^{\log_a x} \implies \log_a y = \log_a x \implies y = x$$

since $\log_a(x)$ is a one-to-one function. Thus, $g(f(x)) = a^{\log_a x} = x$. □

Definition. [The Natural Logarithm] If $a = e$, then we say $\log_a(x) = \ln(x)$. Its inverse is e^x.

4. An Important Conversion

Theorem. *[log conversion] Any positive number a can be written as*

$$a = b^{\log_b a}$$

and especially, take $b = e$, we have $a = e^{\ln a}$.

Proof. So, we ask, what's the power of b so that we get a, i.e., solve $b^x = a$? Taking a log of both sides, we have

$$\log_b b^x = \log_b a \implies x \log_b b = \log_b a \implies x = \log_b a.$$

Thus,

$$a = b^{\log_b a}.$$

□

5. Derivative of the Exponential and Logarithm

Take the following for granted.

Theorem. *If $f(x) = e^x$, then $f'(x) = e^x$.*

Proof. By definition,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x\left(e^h - 1\right)}{h} = e^x \boxed{\lim_{h\to 0} \frac{e^h - 1}{h}}$$

The boxed limit can be shown to take value 1 using the definition of e and the binomial theorem. We omit the details here. □

Theorem. *[Exponential Chain Rule] If $f(x) = e^{g(x)}$, then $f'(x) = e^{g(x)} g'(x)$.*

Proof. The Chain Rule. □

Example. Find the derivatie of $e^{\sqrt{x}}$.

Solution. We apply the Chain Rule. Outer function is e^x and the inner function is $\sqrt{x}$. Thus,

$$\frac{d}{dx}\left[e^{\sqrt{x}}\right] = e^{\sqrt{x}} \frac{d}{dx}\left[\sqrt{x}\right] = \frac{e^{\sqrt{x}}}{2\sqrt{x}}.$$

Example. Find the derivative of $e^{-2\sin x}$.

Solution. We again apply the Chain Rule,

$$\frac{d}{dx}\left[e^{-2\sin x}\right] = e^{-2\sin x} \frac{d}{dx}(-2\sin x) = -2e^{-2\sin x}\cos x.$$

Knowing the derivative for the exponential e^x, we can find the derivative of exponential functions in other bases.

Theorem. *If $f(x) = a^x$, then $f'(x) = a^x \ln a$.*

Proof. Since $a = e^{\ln a}$ by the log conversion theorem, then

$$f(x) = a^x = e^{x \ln a}.$$

Therefore, by the Chain rule, noting that $\ln a$ is just a constant multiplier,

$$f'(x) = e^{x \ln a} \frac{d}{dx}(x \ln a) = a^x \ln a.$$

□

Example. Find the derivative of 2^{x^2}.

Solution. By the Chain Rule, we have the outer function $f(x) = 2^x$ and the inner function $g(x) = x^2$. Thus,

$$\frac{d}{dx}\left[2^{x^2}\right] = f'(g(x))\, g'(x) = \frac{d}{dx}[2^x]\,|_{x^2} \frac{d}{dx}\left[x^2\right] = [2^x \ln 2]_{x^2}(2x) = 2x \cdot 2^{x^2} \ln 2 = x2^{x^2+1} \ln 2.$$

Example. Find the derivative of $2^{e^x \cos x}$.

Solution. Chain rule, with outer function $f(x) = 2^x$ and inner function $g(x) = e^x \cos x$.

$$\begin{aligned}\frac{d}{dx}\left[2^{e^x \cos x}\right] &= f'(g(x))\, g'(x) \\ &= \frac{d}{dx}[2^x]\,|_{e^x \cos x} \frac{d}{dx}[e^x \cos x] \\ &= [2^x \ln 2]\,|_{e^x \cos x}(e^x \cos x - e^x \sin x) \\ &= e^x 2^{e^x \cos x} \ln 2 (\cos(x) - \sin(x))\end{aligned}$$

where we have also applied product rule to $e^x \cos x$.

Furthermore, knowing that the e^x and $\ln x$ are inverses of each other, we can find the derivative of $\ln(x)$ using the Inverse Function Theorem.

Theorem. *If $x > 0$ and $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$. More generally, let g be differentiable. For all values of x for which $g'(x) > 0$, the derivative of $h(x) = \ln(g(x))$ is*

$$h'(x) = \frac{g'(x)}{g(x)}.$$

Proof. We prove the first part. Define $r(x) := f^{-1}(x) = e^x$. Then, by the Inverse Function Theorem, we have

$$f'(x) = \frac{1}{r'(f(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

The second part is a direct application of the Chain Rule. □

Example. Find the derivative of $\ln\left(x^3 + 3x - 4\right)$.

Solution. We apply the second part of the theorem directly with $g(x) = x^3 + 3x - 4$.

$$\frac{d}{dx}\left(\ln\left(x^3 + 3x - 4\right)\right) = \frac{1}{x^3 + 3x - 4} \frac{d}{dx}\left[x^3 + 3x - 4\right] = \frac{3x^2 + 3}{x^3 + 3x - 4}.$$

Example. Find the derivative of $\ln\left(\sqrt{3x+1}\right)$.

Solution. One may be inclined to use the logarithmic chain rule employed in the last example, and one is NOT wrong. But you **must** come to realize that logarithm extracts exponents, effectively, turning the argument into a linear function, which is much easier to differentiate via the Chain Rule. So,

$$y = \ln\left(\sqrt{3x+1}\right) = \ln\left((3x+1)^{\frac{1}{2}}\right) = \frac{1}{2}\ln(3x+1).$$

Then,

$$\frac{dy}{dx} = \frac{1}{2}\frac{3}{3x+1} = \frac{3}{2}\frac{1}{3x+1}.$$

You may try the direct Chain rule, but it's painful.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{\sqrt{3x+1}}\frac{d}{dx}\left(\sqrt{3x+1}\right) \\
&= \frac{1}{\sqrt{3x+1}}\frac{d}{dx}\left((3x+1)^{\frac{1}{2}}\right) \\
&= \frac{1}{\sqrt{3x+1}}\frac{1}{2}(3x+1)^{-\frac{1}{2}}(3) \\
&= \frac{3}{2}\frac{1}{\sqrt{3x+1}}\frac{1}{\sqrt{3x+1}} \\
&= \frac{3}{2}\frac{1}{3x+1}.
\end{aligned}$$

Theorem. *[Derivative of log with bases other than e] If $x > 0$ and $f(x) = \log_b(x)$, then*

$$f'(x) = \frac{1}{x \ln b}.$$

More generally, if $h(x) = \log_b(g(x))$, then for all values of x for which $g(x) > 0$,

$$h'(x) = \frac{g'(x)}{g(x)\ln b}.$$

Proof. Once again, the Inverse Function Theorem. Define $g(x) = f^{-1}(x) = b^x$. Then,

$$f'(x) = \frac{1}{g'(f(x))} = \frac{1}{b^{f(x)}\ln b} = \frac{1}{b^{\log_b x}\ln b} = \frac{1}{x\ln b}.$$

The second part is by the Chain Rule again. $\square$

Example. Find the derivative of $h(x) = \log_4\left(x^2 + \sin x\right)$.

Solution. We apply the theorem above with $g(x) = x^2 + \sin x$ and $b = 4$.

$$h'(x) = \frac{\frac{d}{dx}\left(x^2+\sin x\right)}{(x^2+\sin x)\ln 4} = \frac{2x+\cos x}{(x^2+\sin x)\ln 4}.$$

MATH 1021 LECTURE XV
IMPLICIT DIFFERENTIATION

The objects we have been working with are functions. Their graphs pass the vertical line test. However, for curves such as the circle, there is not a single function that can describe it. Do we just give up and say goodbye to circles? They are so beautiful. Let's not leave beautiful things behind.

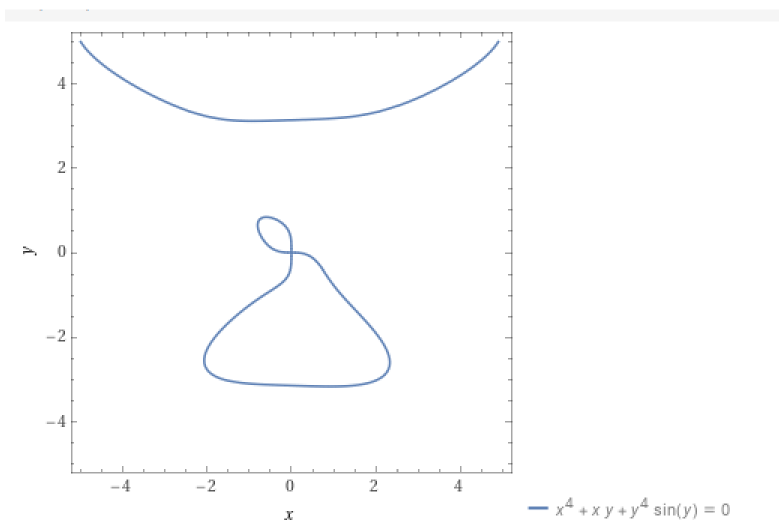
Almost as a habit, we enjoy seeing y and x separated on either side of the equation. Expressions such as

$$y = x^2 + 1$$

are just more satisfying to look at. But more satisfying than what? Consider

$$xy + y^4 \sin y + x^4 = 0.$$

Good luck isolating y and x. But this is still **just** a relationship between x and y. The graph looks like



which is quite cool, like a worm.

We say that the expression

$$y = x^2 + 1$$

defines y **explicitly** in terms of x, while

$$xy + y^4 \sin y + x^4 = 0$$

defines y **implicitly** in terms of x.

Now, for each explicit equation, you can always "convert" it to an implicit equation, namely, the equation

$$y - x^2 = 1$$

defines the function $y = x^2 + 1$ **implicitly.** In other words, implicit equations are more general than functions, and thus act as an umbrella term. In general, an equation defines a function implicitly if the function satisfies that equation. Some implicit equations **can be** turned into explicit equations (such as $y - x^2 = 1$), but some **cannot be** (such as $xy + y^4 \sin y + x^4 = 0$).

Example. Consider the equation $x^2 + y^2 = 25$.

This equation may define **many different functions** implicitly.

(1) $y = \sqrt{25 - x^2}$.

(2) $y = \begin{cases} \sqrt{25 - x^2}, & -5 < x < 0; \\ -\sqrt{25 - x^2}, & 0 < x < 5. \end{cases}$

(3) $y = -\sqrt{25 - x^2}$.

You can verify that all of these functions satisfy the equation $x^2 + y^2 = 25$.

1. Implicit Differentiation

In differentiation of the function $y = f(x)$, we always seek the function $y' = \frac{dy}{dx}$ because we treat y as a function of x. But we cannot always write y as an explicit function of x, i.e., "solve for y". Therefore, finding $\frac{dy}{dx}$ in an implicit setting is useful because we want to understand the slope information in a complicated curve indescribable by the graph of a function. The key is the always write $y = y(x)$.

Example. Assuming that y is defined implicitly by the equation $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

Solution. The previous example has shown that isolating y does NOT give you a **unique** function expressing y explicitly in terms of x. But that's okay. All we are doing is performing $\frac{d}{dx}$ on both sides.

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(25).$$

Now, keep in mind that $y = y(x)$, so

$$\frac{d}{dx}\left(y^2\right) = 2y\frac{dy}{dx}$$

via the Chain Rule. Thus,

$$\begin{aligned} \frac{d}{dx}\left(x^2 + y^2\right) &= \frac{d}{dx}(25) \\ \implies 2x + 2y\frac{dy}{dx} &= 0 \\ \implies \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

given that $y(x) \neq 0$.

Remark. Note that the derivative expression is in terms of both x and y. We were used to seeing it in terms of x only, but in the setting of implicit differentiation, it is not always the case.

Example. Continuing the example above, find the equation of the tangent line at $(3, 4)$.

Solution. We know the derivative is given by

$$\frac{dy}{dx}=-\frac{x}{y}\implies\frac{dy}{dx}\,|_{(x,y)=(3,4)}=-\frac{3}{4}.$$

Then, by point-slope form, we have the equation of the tangent line as

$$y-4=-\frac{3}{4}\left(x-3\right).$$

Example. Continuing the example above again, now find $\frac{d^2y}{dx^2}$, the second derivative.

Solution. We just take one more derivative of $\frac{dy}{dx}$.

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left[\frac{dy}{dx}\right]\\ &= \frac{d}{dx}\left[-\frac{x}{y}\right]\\ &\overset{\text{quotient rule}}{=} -\frac{\frac{d}{dx}\left[x\right]y-x\frac{d}{dx}\left[y\right]}{y^2}\\ &= -\frac{y-x\frac{dy}{dx}}{y^2}\\ &\overset{\text{use }\frac{dy}{dx}=-\frac{x}{y}}{=} -\frac{y-x\cdot\left(-\frac{x}{y}\right)}{y^2}\\ &= -\frac{y+\frac{x^2}{y}}{y^2}\\ &= -\frac{y^2+x^2}{y^3}.\end{aligned}$$

But we do know that $x^2+y^2=25$, so this simplifies even more to

$$\frac{d^2y}{dx^2}=-\frac{y^2+x^2}{y^3}=-\frac{25}{y^3}.$$

Remark. We will learn about what the second derivative implies for the shape of a graph.

Example. Find the equation of the line tangent to thegraph of $y^3+x^3-3xy=0$ at $\left(\frac{3}{2},\frac{3}{2}\right)$.

Solution. Find $\frac{dy}{dx}$ by taking $\frac{d}{dx}$ on both sides of the equation.

$$\begin{aligned}&\frac{d}{dx}\left(y^3+x^3-3xy\right)=0\\ \implies &3y^2\frac{dy}{dx}+3x^2-\frac{d}{dx}\left(3xy\right)=0\\ \implies &3y^2\frac{dy}{dx}+3x^2-\left(3y+3x\frac{dy}{dx}\right)=0\\ \implies &\frac{dy}{dx}\left(3y^2-3x\right)=3y-3x^2\\ \implies &\frac{dy}{dx}=\frac{y-x^2}{y^2-x}.\end{aligned}$$

So, at $(x, y) = \left(\frac{3}{2}, \frac{3}{2}\right)$, we have

$$\frac{dy}{dx}\Big|_{(x,y)=\left(\frac{3}{2},\frac{3}{2}\right)} = \frac{\frac{3}{2} - \left(\frac{3}{2}\right)^2}{\left(\frac{3}{2}\right)^2 - \frac{3}{2}} = -1.$$

By point slope form, we have

$$y - \frac{3}{2} = -\left(x - \frac{3}{2}\right) \implies y = -x + 3.$$

2. Logarithmic Differentiation

We can apply Implicit Differentiation in addition to the Log Conversion Trick learned in the previous lecture, to ugly problems such as finding the derivative of x^x.

Example. $f(x) = x^x$. Find its derivative.

Solution. Please, never write $f'(x) = x \cdot x^{x-1}$ because this quantity simplifies to x^x still. This is **wrong** because the exponent is NOT a constant (as opposed to x^2, or x^3 where the Power Rule applies). Instead, write $y = f(x) = x^x$. Then, taking a log of both sides and appealing to the exponent extraction property of logarithm, we have

$$\begin{aligned} &\ln y = \ln x^x \\ \implies &\ln y = x \ln x. \end{aligned}$$

Now, take $\frac{d}{dx}$ on both sides, i.e., **perform an implicit differentiation**, we find

$$\begin{aligned} &\frac{d}{dx}\left[\ln(y(x))\right] = \frac{d}{dx}\left[x \ln x\right] \\ \implies &\frac{y'(x)}{y(x)} = \left(\ln x + x \cdot \frac{1}{x}\right) \\ \implies &y'(x) = y(x)\left(1 + \ln(x)\right) = x^x\left(1 + \ln(x)\right). \end{aligned}$$

Example. Find dy/dx if $y = \left(2x^4 + 1\right)^{\tan x}$.

Solution. Same trick. Take a log of both sides, we have

$$\ln y = \tan(x) \ln\left(2x^4 + 1\right).$$

Then, take $\frac{d}{dx}$ on both sides, using implicit differentiation,

$$\begin{aligned} &\frac{d}{dx}\left[\ln(y(x))\right] = \frac{d}{dx}\left[\tan(x)\ln\left(2x^4+1\right)\right] \\ \implies &\frac{y'(x)}{y(x)} = \left[\sec^2(x)\ln\left(2x^4+1\right) + \tan(x)\frac{8x^3}{2x^4+1}\right] \\ \implies &y'(x) = \left(2x^4+1\right)^{\tan x}\left[\sec^2(x)\ln\left(2x^4+1\right) + \tan(x)\frac{8x^3}{2x^4+1}\right] \end{aligned}$$

Logarithm has the property that $\ln(xy) = \ln(x) + \ln(y)$ and $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$. Thus, if we have a very ugly but multiplicative expression, such as

$$f(x) = \frac{x\sqrt{2x+1}}{e^x \sin^3(x)},$$

instead of applying the quotient rule to tackle the derivative of f, we can use implicit differentiation via the logarithmic differentiation.

Example. Find $f'(x)$ where $f(x) = \frac{x\sqrt{2x+1}}{e^x \sin^3(x)}$.

Solution. Be my guest if you want to use the Quotient Rule. Email me when you finish or graduate from WPI, whichever comes first.

Since the RHS is a quotient and product of smaller pieces of functions, we should make use of logarithmic differentiation and implicit differentiation.

$$\begin{aligned}
\ln(f(x)) &= \ln\left(\frac{x\sqrt{2x+1}}{e^x \sin^3(x)}\right) \\
&= \ln(x) + \ln\left(\sqrt{2x+1}\right) - \ln(e^x) - \ln\left(\sin^3(x)\right) \\
&= \ln(x) + \frac{1}{2}\ln(2x+1) - x - 3\ln(\sin(x))
\end{aligned}$$

Then, performing $\frac{d}{dx}$ on both sides, we have

$$\begin{aligned}
\frac{d}{dx}\ln(f(x)) &= \frac{d}{dx}\left(\ln(x) + \frac{1}{2}\ln(2x+1) - x - 3\ln(\sin(x))\right) \\
\implies \frac{f'(x)}{f(x)} &= \frac{1}{x} + \frac{1}{2}\frac{2}{2x+1} - 1 - 3\frac{\cos x}{\sin x} \\
\implies f'(x) &= \frac{x\sqrt{2x+1}}{e^x \sin^3(x)}\left[\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x\right]
\end{aligned}$$

MATH 1021 LECTURE XVI
RELATED RATES AND LINEARIZATION

We now take one step further from implicit relationships to ones that relate two simultaneously evolving quantities.

1. Related Rates

Suppose you are pumping a spherical balloon. We care about how the radius of the balloon is changing as a function of time. However, there is no such relationship available to us, to begin with. We do know, on the other hand, a relationship between the volume of a spherical balloon and its radius, namely,

$$V = \frac{4}{3}\pi r^3.$$

We can readily measure how much air we are pumping into the balloon per second. Should this be enough information to calculate how fast the radius is changing per second?

Since both V and r are inevitably changing in time as we pump air into the balloon, we write

$$V(t) = \frac{4}{3}\pi r^3(t).$$

This equation shows that V is a composite function r and t. Thus, to find the rate of change for r against time, we utilize the Chain rule,

$$\frac{dV}{dt} = \left(\frac{4}{3}\pi\right) 3r^2(t)\frac{dr}{dt} = 4\pi r^2(t)\frac{dr}{dt},$$

or

$$V'(t) = 4\pi r^2(t) r'(t),$$

an expression that holds for any time t. This expression further shows that the rate of change of volume depends not only on $r'(t)$, how fast the radius is changing, but also on $r(t)$, the radius at the moment. As you ensure that the air you pump in is of some constant density (something you can control), you can find the rate of change of volume very easily. Therefore, the above relationship is more helpful to determine $r(t)$ ($r(t)$ is also easy enough, just measure it). **This equation is called the equation of related rates**, because we have related the **rates** of $V(t)$ and $r(t)$.

The foremost point of any related rates problems is to be able to construct a model for the time evolution of some physical quantities. Sometimes, you call for geometry knowledge, such as the volume or surface area formula, while sometimes, you are going to think hard about how to set up the correct relationship from given information. Then, to learn about the relationship about the rate of changes, it is merely an application of the chain rule, which is actually the easier part.

Common relationships that would appear in the current scope of related rates are

(1) Volume/Surface Area formula for various shapes – relating volume/surface area to lengths.
(2) Pythagoras – relating lengths and lengths.
(3) Similar triangles – relating lengths to lengths via a common ratio.
(4) Trigonometry – relating angles and lengths.

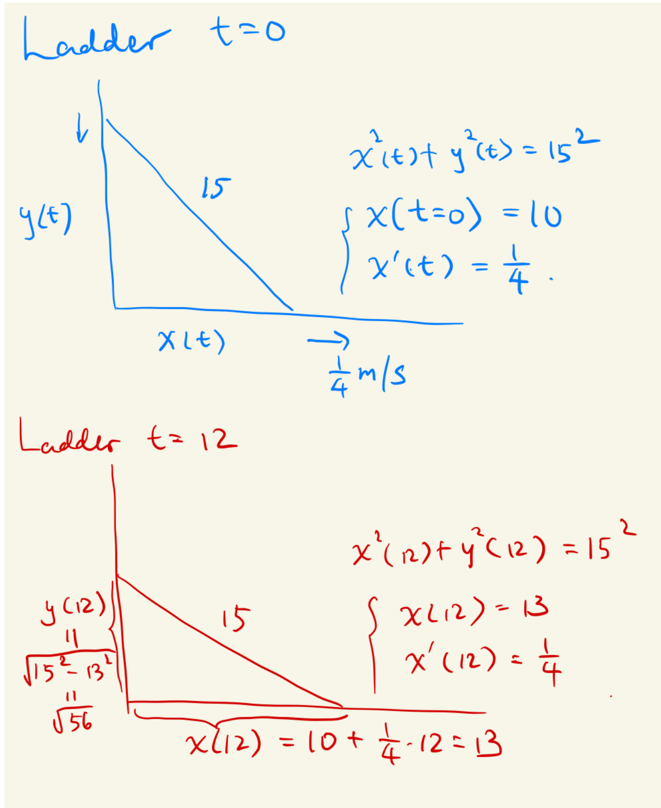
Example. [Volume] Continue the Balloon example. Suppose the rate at which we pump air into the balloon is $2\,\text{cm}^3/\text{s}$. How fast is the radius changing when the radius is 1 cm?

Solution. Guess what, we don't care what time it is. All we know is that at this unknown time t, $r(t) = 1\,\text{cm}$ and $V'(t) = 2\,\text{cm}^3/\text{s}$. Thus, by the equation of related rates,

$$V'(t) = 4\pi r^2(t) r'(t) \implies 2\,\text{cm}^3/\text{s} = 4\pi(1\,\text{cm})^2 r'(t) \implies r'(t) = \frac{2\,\text{cm}^3/\text{s}}{4\pi}\frac{1}{1\,\text{cm}^2} = \boxed{\frac{2}{\pi}\,\text{cm/s}} \approx 0.159\,\text{cm/s}.$$

Example. [Pythagorus] A 15-foot ladder is resting against the wall. The bottom is initially 10 feet away from the wall and is sliding at a constant rate of $1/4\,\text{ft/sec}$. How fast is the top of the ladder moving down the wall 12 seconds after the ladder starts sliding?

Solution. The ladder, vertical wall and the floor form a right triangle. The length relationship here is via Pythagoras.



Denote the floor length and ladder height as $x(t)$ and $y(t)$ respectively. Then we see by Pythagorus that

$$x^2 + y^2 = 15^2$$

since the hypotenuse is always the same ladder – fixed length. As the ladder slides, x and y change over time, so the equation above reads better as

$$[x(t)]^2 + [y(t)]^2 = 15^2.$$

The rate of change in ladder height is precisely $y'(t)$, and we are interested in $y'(12)$. Thus, taking a derivative $\frac{d}{dt}$ of the equation above, we have

$$2x(t)\,x'(t) + 2yy'(t) = 0 \implies x(t)\,x'(t) + y(t)\,y'(t) = 0 \implies y'(12) = -\frac{x(12)\,x'(12)}{y(12)},$$

isolating the unknown.

We know $x'(12) = 1/4\,\text{ft/sec}$. How do we obtain $x(12)$ and $y(12)$? We know initially the bottom is 10 feet away, and it is sliding at a quarter feet per second, so

$$x(12) = 10\,\text{ft} + 12\,\text{sec} \times \frac{1}{4}\,\text{ft/sec} = 13\,\text{ft}.$$

How do we find $y(12)$? Pythagorus,

$$[x(12)]^2 + [y(12)]^2 = 15^2 \implies y(12) = \sqrt{15^2 - 13^2} = \sqrt{56}\,\text{ft}.$$

Now, we have all the ingredient for $y'(12)$,

$$y'(12) = -\frac{x(12)\,x'(12)}{y(12)} = -\frac{13\,\text{ft} \times \frac{1}{4}\,\text{ft/sec}}{\sqrt{56}\,\text{ft}} = \boxed{-\frac{13}{8\sqrt{14}}\,\text{ft/sec}} \approx -0.434\,\text{ft/sec}.$$

The negative sign makes sense because the top of the ladder is sliding **down**.

Example. [Similar Triangles] Water runs into a conical tank at the rate of $9\,\text{ft}^3/\text{min}$. The tank stands point down and has a height of $10\,\text{ft}$ and a base radius of $5\,\text{ft}$. How fast is the water level rising when the water is $6\,\text{ft}$ deep? The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.

Solution. Let's look at each sentence.

"Water runs into a conical tank at the rate of $9\,\mathrm{ft}^3/\mathrm{min}$."

Looking at the unit, volume per minute, it is definitely related to $\frac{dV}{dt}$ or $V'(t)$.

"The tank stands point down and has a height of $10\,\mathrm{ft}$ and a base radius of $5\,\mathrm{ft}$."

Okay, this tells me the dimension of the cone. This not only helps me calculate the total volume, but also informs me how the water grows via its volume formula.

$$V(t) = \frac{1}{3}\pi r^2(t)\,h(t)$$

where

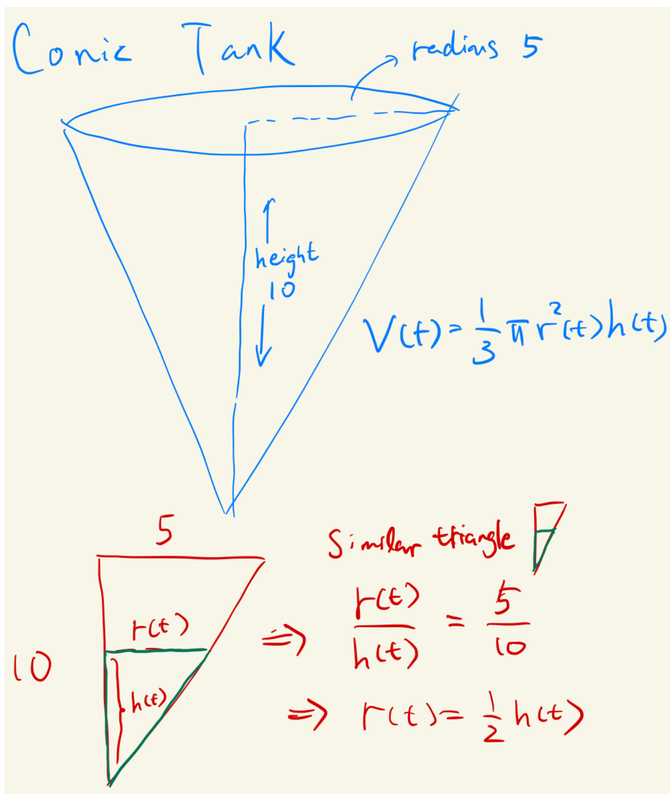
$V(t)$ volume of all water in the tank at time t

$r(t)$ radius of the base of water at time t

$h(t)$ water level at time t

What do we know also? $V'(t) = 9\,\mathrm{ft}^3/\mathrm{min}$, and $h(t) = 6\,\mathrm{ft}$. Hmm, no radius information $r(t)$ nor $r'(t)$ other than its initial dimension. This is trouble because if you proceed to take a derivative $\frac{d}{dt}$ of both sides, we will end up with a product rule on the RHS, containing $r(t)$ and $r'(t)$, both of which we have no clue about.

We need to relate $h(t)$ and $r(t)$, via a brilliant observation due to the geometry of cones



. More precisely, $h(t)$ and $r(t)$, at all times, can be related by similar triangles.

$$\frac{r(t)}{h(t)} = \frac{5}{10} \implies r(t) = \frac{1}{2}h(t).$$

This reduces one unknown, and thus

$$V(t) = \frac{1}{3}\pi\left(\frac{1}{2}h(t)\right)^2 h(t) = \frac{\pi}{12}h^3(t).$$

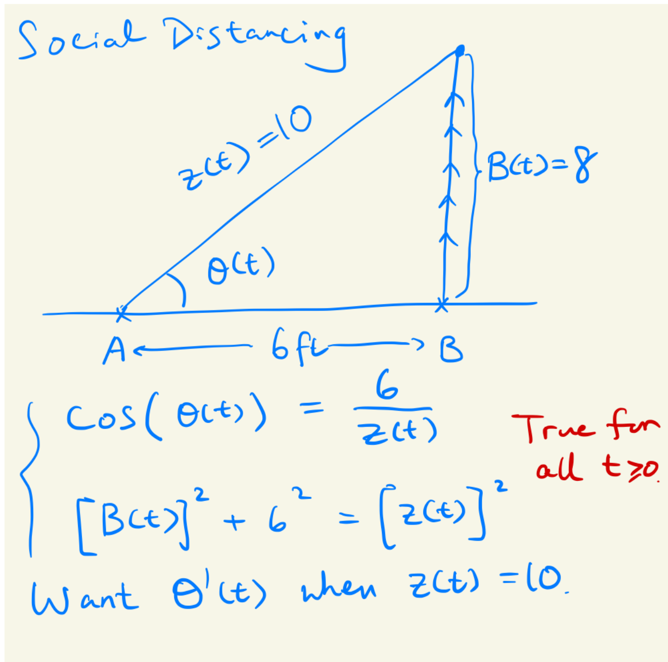
Now, let's proceed with first taking $\frac{d}{dt}$ of both sides to obtain the **equation of related rates**.

$$V'(t) = \frac{\pi}{12}\left(3h^2(t)\right)h'(t) = \frac{\pi}{4}h^2(t)\,h'(t) \implies h'(t) = \frac{4V'(t)}{\pi h^2(t)}.$$

Using the known information, we have

$$h'(t) = \frac{4 \times 9\,\mathrm{ft}^3/\mathrm{min}}{\pi\,(6\,\mathrm{ft})^2} = \boxed{\frac{1}{\pi}\,\mathrm{ft/min}} \approx 0.32\,\mathrm{ft/min}.$$

Example. [Trigonometry] Two people are standing 6 ft apart on the x-axis. One starts walking north at a rate 5 ft/sec, while the other stays put. How fast is the angle between them changing (in radians per second) when the distance between the two is 10 ft?



Solution. So, here, we seek a relationship between the angle and the distance between the two people. By standard trigonometry, we note that

$$\cos(\theta(t)) = \frac{6}{z(t)} \implies \cos(\theta(t))\,z(t) = 6.$$

Therefore, the equation of related rates is

$$\frac{d}{dt}\left[\cos(\theta(t))\,z(t)\right] = 0,$$
$$\implies -\sin(\theta(t))\,\theta'(t)\,z(t) + \cos(\theta(t))\,z'(t) = 0,$$
$$\implies \theta'(t) = \frac{\cos(\theta(t))\,z'(t)}{\sin(\theta(t))\,z(t)} = \frac{z'(t)}{\tan(\theta(t))\,z(t)}.$$

Here, we know that $z(t) = 10\,\mathrm{ft}$. At this moment, the hypotenuse is 10 while the adjacent is 6, so the opposite is $\sqrt{10^2 - 6^2} = 8\,\mathrm{ft}$. This means that

$$\tan(\theta(t)) = \frac{8}{6} = \frac{4}{3}.$$

To find $z'(t)$, we need a relationship between the vertical displacement $B(t)$ of the moving dude and $z(t)$, via Pythagorus

$$[B(t)]^2 + 6^2 = [z(t)]^2 \overset{\frac{d}{dt}}{\implies} 2B(t)\,B'(t) = 2z(t)\,z'(t) \implies z'(t) = \frac{B(t)\,B'(t)}{z(t)}.$$

Now, we know $z(t) = 10$, $B'(t) = 5\,\mathrm{ft/sec}$ and $B(t) = 8$ (the opposite side). Thus,

$$z'(t) = \frac{B(t)\,B'(t)}{z(t)} = \frac{8\,\mathrm{ft} \cdot 5\,\mathrm{ft/sec}}{10\,\mathrm{ft}} = 4\,\mathrm{ft/sec}$$

Altogether

$$\theta'(t) = \frac{z'(t)}{\tan(\theta(t))\,z(t)} = \frac{4\,\mathrm{ft/sec}}{\frac{4}{3} \times 10\,\mathrm{ft}} = \boxed{\frac{3}{10}\,\mathrm{rad/sec}}.$$

This is about 17.19 degrees per second (radian is dimensionless).

2. General Strategy

Moral of the story: zeroth step, there is always hope.

(1) DRAW, DRAW, DRAW A PICTURE.
(2) Name the variables.
(3) Write down the numerical information (such as shape dimensions, like radius, height).
(4) Write down what you are asked to find, in terms of the variables (or their derivatives). It is usually, a rate, so a derivative of some quantity with respect to time.
(5) Write down an equation that relates the variables in step 1. Sometimes, some common sense is needed.
(6) When in doubt, take a derivative.
(7) Evaluate step 4 after finding all the ingredients.

MATH 1021 LECTURE XVII
LINEARIZATION

For a function $f(x)$, one of the goals of differentiation is to <u>learn more about the underlying function $f(x)$</u>. Learning more about the underlying function $f(x)$ then helps us make predictions using $f(x)$ either more accurately or more efficiently.

However, there are certain functions that are costly to numerically evaluate, e.g.,

$$f(x) = \sin(x)\, e^{\cos^2(x)}.$$

Evaluating $f(x)$ involving taking the sine, exponential and cosine, all of which are **nonlinear functions**, in other words, they are not as easy as just multiplying and adding. Imagine doing this calculation 10 million times – saving a few milliseconds in **each evaluation** here is worth a lot of time.

A method always depends on the specific task at hand. Suppose the task is to find the values of $f(x)$ on $\left[0, \frac{\pi}{2}\right]$, and we want to avoid using $f(x)$ directly to save some computation time. Of course, if you are not evaluating $f(x)$ directly, you pay some price of approximation error.

1. Linearization

One such approximation is called **the linearization of** $f(x)$ **at** $x = a$ **for** $a \in \left[0, \frac{\pi}{2}\right]$. "$x = a$" is often referred to as the **center of the linearization**.

The Linearization of $f(x)$ at $x = a$ can also be referred to as "**The Linear Approximation of** $f(x)$ **at** $x = a$".

1.1. Approach 1: System of Equations.

Define $L(x)$, a line of the form $L(x) = b + mx$. We want to find the slope m and the intercept b such that

$$\begin{aligned} f(a) &= L(a), \\ f'(a) &= L'(a), \end{aligned}$$

that is, the line $L(x)$ has the same slope as $f(x)$ at $x = a$ and goes through the same point $(a, f(a))$. We have two unknowns, and two equations. We find

$$\begin{aligned} f(a) = L(a) &\implies f(a) = ma + b, \\ f'(a) = L'(a) &\implies m = f'(a), \end{aligned}$$

so

$$b = f(a) - f'(a)\, a.$$

Altogether,

$$f(x) \approx \boxed{L_{x=a}(x)} = f(a) - f'(a)\, a + f'(a)\, x \boxed{= f(a) + f'(a)(x - a)}.$$

Note that a subscript is put under L, just to remind you that you are linearizing about the point $x = a$. The linearization WILL change as a changes.

1.2. Approach 2: Point-slope Form.

On the surface, this procedure produces precisely the equation of the tangent line. Writing $y = L(x)$, we recognize that

$$y = L_{x=a}(x) = f(a) + f'(a)(x - a) \iff y - f(a) = f'(a)(x - a)$$

is precisely the point-slope form of the tangent line at $(a, f(a))$ with slope $f'(a)$. So, you may say that the two approaches are equivalent

$$\boxed{\text{Find the linearization } L_{x=a}(x) \text{ of } f(x) \text{ at } x = a} \iff \boxed{\text{Find the equation of the tangent line at } (a, f(a))}$$

because they produce the same result.

1.3. **Difference between The Two Approaches.** Even though the two approaches produce the same result, they are conceptually different.

(1) A tangent line is a line that touches the function exactly at one point. It is a mathematical object (that lives in space, 1D, 2D, 3D ...).
(2) Linearization of a function is using a line to approximate a function near the point of interest. It is a mathematical concept carried out by a mathematical procedure.

The story of the equation of the tangent line ends here becaues it is after all just a line. But the story of linearization extends to higher dimensions. It is the modern workhorse of solving nonlinear problems.

1.4. **Examples.**

Example. Find the linearization $L_{x=1}(x)$ of $f(x)=x^2$ at $x=1$. Approximate $f(1.01)$ by calculating $L_{x=1}(1.01)$. Calculate the relative error $err_{\text{relative}}=\left|\frac{L_{x=1}(1.01)-f(1.01)}{f(1.01)}\right|$ to see that the difference is not big.

Solution. To find the linearization, you may choose either approach from above.

$$L_{x=1}(x)=f(1)+f'(1)(x-1)=1^2+2(x-1)=2x-1.$$

We find that $L_{x=1}(1.01)=2\cdot 1.01-1=2.02-1=1.02$. The truth yields $f(1.01)=1.0201$. So

$$err_{\text{relative}}=\left|\frac{L_{x=1}(1.01)-f(1.01)}{f(1.01)}\right|=\left|\frac{1.02-1.0201}{1.0201}\right|\approx 9.8\times 10^{-4}.$$

We did pretty good here.

Example. Use the same linearization to approximate $f(1.98)$.

Solution.

$$L_{x=1}(1.98)=2\cdot 1.98-1=3.96-1=2.96.$$

However, $f(1.98)=3.9204$, so

$$err_{\text{relative}}=\left|\frac{L_{x=1}(1.98)-f(1.98)}{f(1.98)}\right|=\left|\frac{2.96-3.9204}{3.9204}\right|\approx 0.245=24.5\%.$$

We missed by A LOT, unless you have low standards.

Example. Which point should we then perform the linearization of $f(x)=x^2$ to properly approximate $f(1.98)$?

Solution. We should, of course, choose a point close to $x=1.98$, e.g., $a=2$.

$$L_{x=2}(x)=f(2)+f'(2)(x-2)=4+4(x-2)=4x-4.$$

Then,

$$L_{x=2}(1.98)=4\times 1.98-4=7.92-4=3.92.$$

Thus,

$$err_{\text{relative}}=\left|\frac{L_{x=2}(1.98)-f(1.98)}{f(1.98)}\right|=\left|\frac{3.92-3.9204}{3.9204}\right|\approx 1.02\times 10^{-4},$$

which is again pretty good.

Example. Find the linearization of $g(x)=\sqrt{x+1}$ at $x=0$.

Solution.

$$L_{x=0}(x)=g(0)+g'(0)(x-0).$$

We find

$$g(0)=\sqrt{0+1}=1$$
$$g'(x)=\frac{1}{2\sqrt{x+1}}\implies g'(0)=\frac{1}{2\sqrt{0+1}}=\frac{1}{2}.$$

Thus,

$$L_{x=0}(x)=g(0)+g'(0)(x-0)=1+\frac{x}{2}.$$

How good is this approximation?

x	Approximation $L_{x=0}(x) = 1 + \frac{x}{2}$	True Value $g(x) = \sqrt{x+1}$	Relative Error $\left\lvert \frac{L_{x=0}(x) - g(x)}{g(x)} \right\rvert$
0.005	$1 + \frac{0.005}{2} = 1.0025$	$\sqrt{1.005} \approx 1.002497$	$\left\lvert \frac{1.0025 - \sqrt{1.005}}{\sqrt{1.005}} \right\rvert = 3.11 \times 10^{-6}$
0.05	$1 + \frac{0.05}{2} = 1.025$	$\sqrt{1.05} \approx 1.024695$	$\left\lvert \frac{1.025 - \sqrt{1.05}}{\sqrt{1.05}} \right\rvert = 2.98 \times 10^{-4}$
0.2	$1 + \frac{0.2}{2} = 1.1$	$\sqrt{1.2} \approx 1.095445$	$\left\lvert \frac{1.1 - \sqrt{1.2}}{\sqrt{1.2}} \right\rvert = 4.16 \times 10^{-3}$
3	$1 + \frac{3}{2} = 2.5$	$\sqrt{4} = 2$	$\left\lvert \frac{2.5 - \sqrt{4}}{\sqrt{4}} \right\rvert = 0.25 = 25\%$

As you can see, the approximation becomes quite bad when we venture away from $x = 0$.

MATH 1021 LECTURE XVIII
DIFFERENTIALS

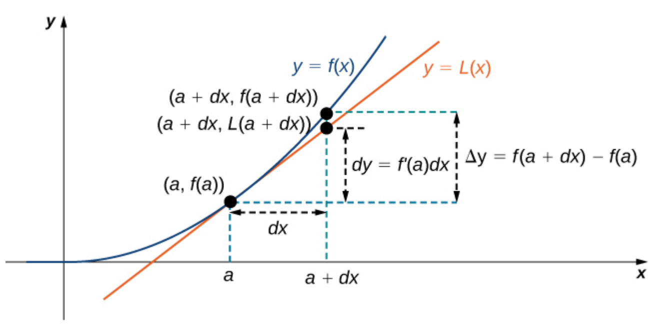
1. Motivation

Just now, we found that the point at which we linearize affects the accuracy of the linear approximation. So, we choose a point closer to the point of interest at which we linearize. Of course, this still incurs, though smaller, error. From the example on the linearization of $f(x) = x^2$, we find that the linearization at $x = 1$ provides us a good approximation for 1.01^2, while the linearization at $x = 2$ does a decent job for 1.98^2.

Here comes a higher level question: for the same function, do approximation errors depend on the point at which we linearize? In the context of the example for $f(x) = x^2$, is the approximation error from $L_{x=1}(x)$ to estimate 1.01^2 the same as that from $L_{x=2}(x)$ to estimate 1.98^2? If not, what does this error depend on?

More simply put, what is the amount of change in output when the input changes by a little bit?

One picture says it all.



Linearization of a function $f(x)$ at $x = a$ is given by

$$L(x) - f(a) = f'(a)(x - a)$$

where we put $f(a)$ on the LHS intentionally. Thus, $(x - a)$ measures how much x deviates from a, where x is the point of interest; in response to this deviation, the change in function values can be **estimated** by $L(x) - f(a)$ (exact value is $f(x) - f(a)$). The response is carried out by precisely the multiplier, i.e., the slope $f'(a)$.

Our goal below is to <u>relate changes in x to changes in y, namely, $L(x) - f(a)$ v.s. $(x - a)$</u>. The following could be a little technical, even with the help of the graph. For applications and examples, you may skip to the next section.

Definition. Let $y = f(x)$ be a differentiable function. The differential dx is an independent variable. The differential dy (a dependent variable) is given by

$$dy(x, dx) = f'(x)\,dx.$$

Note that dy is a function of x AND dx.

The differential dx is a deep concept. It shows infinitesimal change in x-values a very very small change, almost negligible. But when paired with the instantaneous rate of change, i.e. the derivative, it produces also an infinitesimal change in y. dx is different from Δx, with the former being a sizeable change in x-values, which also leads to a sizeable change in y-values, that is, Δy.

$$\Delta y = f(a + \Delta x) - f(a).$$

The corresponding change in the linearization $L_{x=a}$ is

$$\begin{aligned}\Delta L &= L(a + \Delta x) - L(a) \\ &= f(a) + f'(a)[(a + \Delta x) - a] - f(a) \\ &= f'(a)\Delta x.\end{aligned}$$

By definition, this is precisely, $dy(a, dx)$ if $x = a$ and $dx = \Delta x$. Therefore, dy represents the amount the linearization rises or falls when x changes by an amount $dx = \Delta x$. Thus, $dy = \Delta L$ when $dx = \Delta x$.

In other words,

$$f(a + dx) \approx f(a) + \Delta L = f(a) + \boxed{dy} = f(a) + \boxed{f'(a)\,dx}, \tag{1.1}$$

that is, we can approximation a function value $f(a + dx)$ with the knowledge of $f(a)$, $f'(a)$ and the horizontal distance dx between a known point a and a point of estimation $a + dx$.

2. Examples

In practice, we just want to make simpler statements.

Example. We know $\sqrt{4}$ very well. Can we use it to estimate $\sqrt{4.02}$?

Solution. You may use linearization of the function $f(x) = \sqrt{x}$ at $x = a = 4$. Let's use differentials here. We want to estimate

$$f(4.02) = f(4 + 0.02)$$

using the strategy in Equation 1.1.

From that differential formula, we need $f'(x) = \frac{1}{2\sqrt{x}}$, and identify that $a = 4$ and $dx = 0.02$. Thus, we calculate the boxed item in that formula,

$$dy = f'(x)\,dx \implies dy = \frac{1}{2\sqrt{x}}\,|_{x=4}\,(0.02) = \frac{1}{2\sqrt{4}}(0.02) = 0.005.$$

Thus,

$$\sqrt{4.02} = f(4 + 0.02) \approx f(4) + dy = \sqrt{4} + 0.005 = 2.005.$$

The true answer $\sqrt{4.02} \approx 2.004994$, so we did pretty good.

Example. Given an industrial standard for a cylindrical screw's radius, say $r = 0.5\,\text{cm}$ and a height of $h = 10\,\text{cm}$ how much error in manufacturing this radius is allowed so the change in volume due to this error is within 5%? One may write the volume of the cylindrical screw as a function of the radius r,

$$V(r) = \pi r^2 h.$$

The question is simple and practical. Let's calculate the differential of V in terms of r and dr.

$$dV\left(r, dr\right) = 2\pi r h dr.$$

Now, we identify that $r = 0.5$, $h = 10$. Furthermore, we require that

$$\frac{dV}{V} < 0.05$$

where we know the formula for both dV and V. We need to find how this relates to the relative error in radius, $\frac{dr}{r}$.

$$0.05 > \frac{dV}{V} = \frac{2\pi r h dr}{\pi r^2 h} = 2\frac{dr}{r} \implies \frac{dr}{r} < 0.025 = 2.5\%.$$

Thus, in order to keep the relative error in volume under 5%, the manufacturing error in radius should be kept below 2.5%. Note that h is irrelevant here since we assumed that there is no error in height manufacturing.

Example. The radius r of a circle increases from $a = 10$ m to 10.1 m. Use dA, the area differential, to estimate the increase in the circle's area A. Estimate the area of the enlarged circle and compare your estimate to the true area found by a direction calculation.

Solution. Since $A = \pi r^2$, then

$$dA\left(r, dr\right) = 2\pi r dr = 2\pi\left(10\right)\left(0.1\right) = 2\pi \,\mathrm{m}^2.$$

Therefore,

$$A\left(r + dr\right) \approx A\left(r\right) + dA = \pi\left(10\right)^2 + 2\pi = 102\pi \,\mathrm{m}^2,$$

i.e., the area of a circle of radius 10.1 m is approximately $102\pi \,\mathrm{m}^2$.

By a brute force calculation, we find that $A\left(10.1\right) = \pi\left(10.1\right)^2 = 102.01\pi \,\mathrm{m}^2$. So, with our differential estimate, we incur an error of $0.01\pi \,\mathrm{m}^2$, which is NOT bad at all.

The true difference is

$$\Delta A = A\left(10.1\right) - A\left(10\right) = \pi\left(10.1\right)^2 - \pi\left(10\right)^2 = \pi\left(102.01 - 100\right) = 2.01\pi\, m^2,$$

and our differential estimate error is

$$dA = 2\pi \,\mathrm{m}^2$$

which is obviously smaller.

Example. Use differentials to estimate $\left(7.97\right)^{1/3}$.

Solution. The function we use here is $f\left(x\right) = x^{1/3}$, and we need to hone in on $a = 8$ (a sensible choice, though you can pick 8.5 but you are just making life hard).

Note that $dx = -0.03$; it is the displacement from the center $a = 8$ to the point of estimate 7.97. We also need derivative information at $a = 8$.

$$f'\left(x\right) = \frac{1}{3}x^{-\frac{2}{3}} \implies f'\left(8\right) = \frac{1}{3}\left(8\right)^{-\frac{2}{3}} = \frac{1}{3}\frac{1}{\left(8^{\frac{1}{3}}\right)^2} = \frac{1}{3}\frac{1}{4} = \frac{1}{12}.$$

So, by the differential formula, we have

$$\left(7.97\right)^{1/3} = \left(8 - 0.03\right)^{1/3} = f\left(8 - 0.03\right) \approx f\left(8\right) + dy = f\left(8\right) + f'\left(8\right)dx = 2 + \frac{1}{12}\left(-0.03\right) = 1.9975.$$

The true value (via a calculator) is $\left(7.97\right)^{1/3}$ is 1.997497 approximately, so we are accurate up to 6 decimals, which is pretty impressive.

MATH 1021 LECTURE XIX
FUNCTION EXTREMA

One important application of the derivative is to find the optimal solutions to problems. An optimal solution maximises or minimises a certain performance metric, such as total cost of a complicated business strategy, utility function of an economic model, or the amount of drugs delivered in medicine under spurious environment. Many physical questions can be turned into a mathematical one.

First, we must define the **absolute maximum and minimum**, particularly, of the values of a function.

Definition. [Absolute Max/Min] Let f be a function with domain D. Then f has an **absolute maximum (or global maximum)** value on D at a point c if

$$f(x) \leq f(c) = M < \infty, \quad \text{for all } x \in D;$$

and an **absolute minimum (or global minimum)** value on D at a point c if

$$f(x) \geq f(c) = m > -\infty, \quad \text{for all } x \in D.$$

Remark. M and m are finite numbers. They don't have to exist, i.e., absolute maximum and minimum don't have to exist on every domain D.

Maximum and minimum values of f are called **extreme values or extrema** of f.

Example. Consider the function $f(x) = x^2$ on several domains.

D	**Global Extrema**
$(-\infty, \infty)$	No global extrema because the function is unbounded; global minimum at $(0, 0)$
$[0, 2]$	Global maximum at $(2, 4)$; global minimum at $(0, 0)$
$(0, 2]$	Global maximum at $(2, 4)$; no global minimum
$(0, 2)$	No global extrema.

Remark. **Function extrema depends on the domain.**

Note that the existence of global extrema depends on the domain. When the interval is open on one or both ends, the existence of global extrema becomes a bit more subtle. For example, in the third case, $(0, 2]$, since $x = 0$ is not included, the x-values are only getting so close to 0 without touching it. This means, if you claim that there is a global minimum achieved by some c near 0, by a property of the real number system, you can also find another number $d < c$ such that $d^2 < c^2$, which defeats c as a global minimum. By this argument, you won't attain a global minimum!

However, you note that on the closed interval $[0, 2]$, we have both extrema. This in fact is a true statement if we further suppose that the function is continuous and the interval is bounded, in addition to being closed.

Theorem. *[Extreme Value Theorem] If f is continuous on $[a, b]$, then f attains both a global maximum value M and a global minimum m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ such that $f(x_1) = m$ and $f(x_2) = M$, and*

$$m \leq f(x) \leq M$$

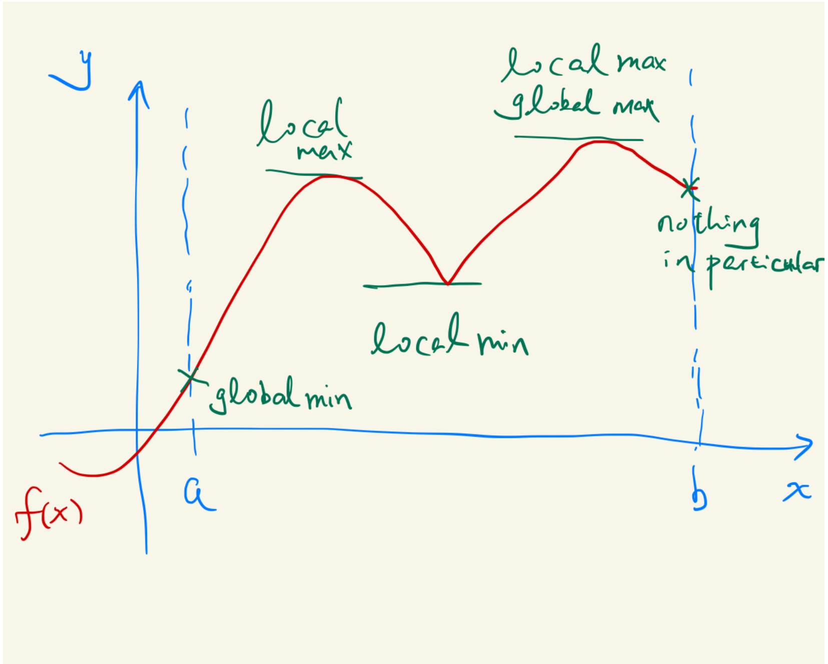
for every other $x \in [a, b]$.

Remark. The requirements that the domain be closed and bounded and the function be continuous are essential for the conclusions to hold. If you lose any one of them, then the conclusion needs not hold. See the 3rd and 4th case in the previous example, where the function is continuous but the interval is not closed (though bounded).

Example. If the interval is closed and bounded, but the function is not continuous, the conclusion doesn't hold. Consider

$$f(x) = \begin{cases} x, & 0 \leq x < 1; \\ 0, & x = 1. \end{cases}$$

The function is continuous at every point on $[0, 1]$ but never achieves a global maximum.



Definition. A function f has a **relative maximum (local maximum)** value at a point c within its domain D if

$$f(x) \leq f(c)$$

for all $x \in D$ in *some* open interval containing c. A function f has a **relative minimum (local minimum)** value at a point c withint its domain D if

$$f(x) \geq f(c)$$

for all $x \in D$ in *some* open interval containing c.

The beauty here about local extrema is that you just need to find **one** neighborhood containing yourself such that you are optimal in this very neighborhood. The figure says it all – no smaller/greater value of f **nearby**.

MATH 1021 LECTURE XX
FIRST DERIVATIVES AND THEIR INFERENCES

Finding extremas is one necessary step to determine the optimal solutions to applications. First, we have a necessary condition for local extremas.

1. First Derivative Theorem and Critical Points

Theorem. *(First derivative theorem/Fermat's Theorem)* ***If*** *a differentiable f has a local maximum or minimum value at an interior point c of its domain, and if f is defined at c,* ***then*** $f'(c) = 0$.

Remark. The way to read this Theorem is important. The theorem suggests that you have **already** identified the local extrema, then the derivative at this extrema must vanish. In other words, seeing a priori a vanished derivative does NOT guarantee that the point is a local extrema (it may be, but is not guaranteed).

The remark motivates the **contrapositive** of the theorem. A general theorem involves a one directional implication, for example, "if A then B". The contrapositive of this statement is "if not B then not A", a logically equivalent statement.

Theorem. *(Contrapositive statement of the First Derivative Theorem, namely,* ***"if not*** B ***then not*** A*")*
If *$f'(c)$ is defined but $f'(c) \neq 0$ where c is an interior point of the domain of f,* ***then*** *f cannot achieve a local maximum or minimum at $x = c$.*

However, the theorem does not tell us how to sufficiently qualify a point as a local extrema. It only tells us, if it is one, then it **necessarily** has zero slope. Is the converse true? Apparently not.

Example. Consider the function $f(x) = x^3$. We find that $f'(x) = 3x^2$, which means $f'(0) = 0$. But looking at the graph of x^3, we find that $x = 0$ is neither a local max nor a local min (just draw a neighborhood around it).

Do we give up on the points such that $f'(x) = 0$ altogether then? No, we never give up. They are still valuable because they **could** be local maximum or minimum. We just need to do more work to verify that they **are** local extrema. In fact, there are three possible places where extreme values can be achieved (global or local):

(1) Interior points where $f' = 0$.
(2) Interior points where f' is undefined.
(3) Endpoints of the domain – this is due to the Extreme Value Theorem.

Items 1 and 2 seem to be important enough to warrant a mathematical term.

Definition. [Critical point] An interior point of the domain of a function f where f' is **zero** or **undefined** is a **critical point** of f.

Remark. Once again, being a critical point does NOT guarantee that it is an extreme value.

Theorem. *[Absolute Extrema for Differentiable Functions]*
If f is a differentiable function on some open interval (a, b) and continuous on $[a, b]$, then f achieves its absolute extrema at either the endpoints or at the critical points.

Remark. Compare this to the Fermat's theorem which only requires continuity, being differentiable means we just need to look at a **few things**, not everything in the interval.

The above definitions and theorems give us a procedure to find the global extreme values of f on a given interval $[a, b]$.

(1) Find all critical points of f on the interval, i.e., form a pool of candidates.
(2) Evaluate f at the critical points and endpoints.
(3) Largest value is global max, while smallest value is global min.

Example. Find the absolute max and min of $g(t) = 8t - t^4$ on $[-2, 1]$.

Solution. We follow the steps precisely.

(1) Critical points.
First, we find that $g'(t) = 8 - 4t^3$. Setting it equal to 0, we find that a critical point satisfies
$$0 = 8 - 4t^3 \implies t^3 = 2 \implies t = 2^{\frac{1}{3}}.$$
However, note that $2^{\frac{1}{3}} > 1$ which means $2^{\frac{1}{3}} \notin [-2, 1]$. Therefore, there exists NO critical points in $[-2, 1]$.

(2) Evaluation of critical points and the endpoints.
So, we just care about the endpoints $t = -2$ and $t = 1$.
$$g(-2) = 8 \cdot (-2) - (-2)^4 = -16 - 16 = -32.$$
$$g(1) = 8 \cdot (1) - (1)^4 = 7.$$

(3) Compare.
We find that $(1, 7)$ is the absolute maximum, and $(-2, -32)$ is an absolute minimum.

Example. [Practical Example] The range R of a projectile launched on the ground, i.e. horizontal distance travelled, satisfies
$$R(\theta) = \frac{v_0^2}{g} \sin(2\theta)$$
where θ is the launch angle, v_0 is initial velocity and g is gravitational acceleration constant. Is there a launch angle such that the range is maximized?

Solution. First, note that $\theta \in [0, \pi]$ (be realistic), and we need to find the optimal θ on this interval.
First, find the critical points:
$$\frac{dR}{d\theta} = \frac{v_0^2}{g} 2\cos(2\theta) = 0 \implies 2\theta = \frac{\pi}{2} \implies \theta = \frac{\pi}{4}.$$
The problem is NOT done here because this critical point is only a candidate for local extrema, not a guarantee.
We must evaluate the critical point and the endpoints of the interval to check for absolute extrema.

2. Motivation to study Monotone Functions: Free Space Optimization

What if the domain is unbounded? We can still find critical points, but we need to examine the sign of the derivative in the neighborhood of these critical points to make a judgement on whether that critical point is a local max or min. A very general statement about the sign of f' goes as follows,
$$\begin{aligned} f' &> 0 \quad \text{increasing} \\ f' &< 0 \quad \text{decreasing} \end{aligned}$$
which comes particularly handy when one is tasked to find the optimum of a function on unbounded domains (where you got no endpoints to check). Functions with these properties are worth a separate section. Before, we state an important results in Differential Calculus about the slopes of a differentiable function.

3. The Mean Value Theorem

From the Intermediate Value Theorem, we've learned that a continuous function achieves every value between $f(a)$ and $f(b)$ on $[a, b]$. So, what can we say about a differentiable function and its slope value on the same interval? The observation is a geometric one. This result helps us justify why $f' > 0$ implies that the function is increasing (though it seems obvious).

Theorem. *[Mean Value Theorem] If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a number $c \in (a, b)$ such that*
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$
In other words, somewhere on this interval, the function f achieves a slope equal to that of the secant line between $(a, f(a))$ and $(b, f(b))$.

The next result is a direct consequence of the mean value theorem, though the result itself may seem trivial, yet it is not.

Corollary 1. *If $f'(x) = 0$ at every point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.*

Proof. Note that f satisfies the hypothesis of the mean value theorem. We wish to show $f(x_1) = f(x_2)$ with $x_1 \neq x_2$ for **any** x_1, x_2 that lie in (a, b).

Without loss of generality, assume $x_1 < x_2$ [and if not, then reverse the label of x_1 and x_2]. By the Mean Value Theorem, there must be a point $c \in (x_1, x_2)$ such that

$$0 = f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \implies f(x_1) = f(x_2).$$

□

When two functions have the same derivative, can we relate the two functions themselves?

Corollary 2. *If $f'(x) = g'(x)$ at every point $x \in (a, b)$, then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$.*

Solution. Define $h(x) = f(x) - g(x)$. We note that $h'(x) = 0$. Then by the previous corollary, $h(x) = C \implies f(x) = g(x) + C$, and we are done.

Example. [A precursor to antidifferentiation] Suppose $f(0) = 5$ and $f'(x) = 2$ for all x. What is $f(x)$?

Solution. We assume that we have no integration knowledge but only the two corollaries above. We don't know what $f(x)$ looks like.

We note that a function $g(x) = 2x$ satisfies $g'(x) = 2$ which matches $f'(x)$. Thus, by the previous Corollary,

$$f(x) = g(x) + C.$$

To find C, we use the condition $f(0) = 5$, i.e.,

$$5 = f(0) = g(0) + C = 2 \cdot 0 + C \implies C = 5.$$

Altogether

$$f(x) = 2x + 5.$$

4. Monotone Functions and First Derivative Test

The local behaviour of a function is either increasing or decreasing. Staying constant is boring (and easy to analyze, see Corollary 1 above). Therefore, it is useful to mathematically characterize a function that is purely increasing or decreasing.

Definition. We say that f is **monotone increasing** on an interval I if we have $f(x_1) < f(x_2)$ (resp. **monotone decreasing** with $>$), for every $x_1 < x_2$ on I.

Corollary 3. *If f is continuous on $[a, b]$ and differentiable on (a, b), and if $f'(x) > 0$ for every $x \in (a, b)$, then f is **monotone increasing** (resp. < 0, **monotone decreasing**).*

Proof. This result, again, is a direct consequence of the Mean Value Theorem. For any two points $x_1 < x_2$ on (a, b), there is some point c such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Now, since $f'(x) > 0$ for every $x \in (a, b)$, then $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$. Since $x_2 > x_1$ by assumption, this suggests that $f(x_2) > f(x_1)$, proving that f is **monotone increasing** (by definition). □

Theorem. *[First Derivative Test] At a critical point $x = c$,*

1. *"$\cup$" : If f' changes from $-$ to $+$ at $x = c$, then $x = c$ is a **local minimum**.*
2. *"$\cap$":If f' changes from $+$ to $-$ at $x = c$, then $x = c$ is a **local maximum**.*
3. *If f' does not change sign through c, then we can claim **nothing** at $x = c$. [Think about $f(x) = x^3$ at $x = 0$]*

Example. Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing and on which f is decreasing.

Solution. We find the critical points first. $f'(x) = 3x^2 - 12$. Setting the derivative equal to 0 yields

$$3x^2 - 12 = 0 \implies x = \pm 2.$$

We then create nonoverlapping intervals $(-\infty,-2)$, $(-2,2)$ and $(2,\infty)$. We plug in convenient values from each interval to check the sign of f',

$$\begin{aligned} f'(-3) &= 3(-3)^2 - 12 = 15 > 0 \\ f'(0) &= -12 < 0 \\ f'(3) &= 3(3)^2 - 12 = 15 > 0. \end{aligned}$$

Therefore, by the first derivative test,

$$\begin{aligned} x &= -2 \text{ local min} \\ x &= 2 \text{ local max} \end{aligned}$$

5. Free Space Optimization: Examples

Example. Identify the domain of $y = x^{2/3}(x+2)$ and find its local max and min.

Solution. The domain is $(-\infty,\infty)$.

(1) We first locate the critical points.

$$y' = \frac{2}{3x^{1/3}}(x+2) + x^{2/3} = \frac{1}{x^{1/3}}\left(\frac{2}{3}(x+2)+x\right).$$

Setting it equal to zero, we find

$$\frac{2}{3}(x+2) + x = 0 \implies x = -\frac{4}{5}.$$

Also, critical points also include the points such that y' is undefined. Thus, we must also include $x = 0$.

(2) Evaluate – and this problem does NOT have endpoints. Do NOT evaluate at $x = \pm\infty$.

$$y\left(-\frac{4}{5}\right) = \frac{12}{15}10^{1/3} > 0$$

and $y(0) = 0$. Does this suggest that $x = -\frac{4}{5}$ is a local max and $x = 0$ is a local min? No. We got more work to do, though these checkpoints are very useful.

(3) Study the sign of the derivative near these two critical points.

Critical points/intervals	$x < -\frac{4}{5}$	$x = -\frac{4}{5}$	$-\frac{4}{5} < x < 0$	$x = 0$	$x > 0$
$f'(x)$	$+$	0	$-$	undefined	$+$
$f(x)$	increasing	**local max**	decreasing	**local min**	increasing

where we bolded the local max and min here – they are filled AFTER you find the signs of f' in the neighborhood of the critical points.

To check the sign of the first derivative on various intervals, we plug in a typical value in that interval (hopefully convenient for evaluation), and find out the sign. If it is positive, it means that the function is increasing, and decreasing otherwise. When a function increases up to a critical point, and decreases past it, then the critical point is a local max. Similarly, if a function decreases down to a critical point, and increases past it, then the critical is a local min.

Example: Speeding Cars!

Another interpretation is that the average rate of change must be equal to the instantaneous rate of change somewhere between. In fact, many real world problems can lead to nice conclusions using the mean value theorem. We see its usage in detecting speeding cars!

Example. This very fact also explains how electronic cameras can decide whether a car has sped or not. Consider two cameras placed 1 mile apart on a highway with speed limit 60 miles/hour. Now, consider time t as our independent variable and car position $x(t)$ as our dependent variable. Denote t_1 and t_2 as the time the car reaches camera 1 and 2 respectively. Suppose x is a continuous function of t on $[t_1,t_2]$ and differentiable on (t_1,t_2), then we have by mean value theorem that there exists some time $s \in (t_1,t_2)$ such that

$$v(s) = x'(s) = \frac{x(t_2) - x(t_1)}{t_2 - t_2} = \frac{1\,\text{mile}}{(t_2-t_1)\ \text{seconds}}.$$

The camera will have access to the time difference $t_2 - t_1$ (in seconds). Therefore, it is able to compute $\frac{1}{t_2-t_1}$. Now, in units of miles per hour, we write

$$(t_2 - t_1)\ \text{seconds} = \frac{(t_2 - t_1)\ \text{seconds}}{3600\ \text{seconds/hour}}$$

and thus if we find

$$\frac{3600\,\text{miles}}{(t_2 - t_1)\ \text{hours}} > 60\,\text{mph},$$

then the car is guaranteed to have sped at some point between the two cameras.

6. A synopsis: General Strategy

6.1. **Absolute Extrema.**

(1) Find all critical points of f on the interval, i.e., form a pool of candidates.
(2) Evaluate f at the critical points and endpoints.
(3) Largest value is global max, while smallest value is global min.

6.2. **Local Extrema.**

(1) Find all critical points of f on the interval, i.e., form a pool of candidates.
(2) Draw the number line by labeling the critical points.
(3) Study the sign of f' on each interval separated by the critical points.
(4) Apply 1st derivative test in Section 4.

MATH 1021 LECTURE XXI
SECOND DERIVATIVES AND FUNCTION CONVEXITY

1. Second Derivatives and Function Convexity

There are two ways of increasing, $\curlyvee$ or $\curlywedge$. The difference here is how it curves; in other words, how the curve faces down or up.

Definition. The graph of a differentiable function $y = f(x)$ is

(1) **convex (concave up)** on an open interval I if f' is increasing on I.
(2) **concave (concave down)** on an open interval I if f' is decreasing on I.

Theorem. *If f'' exists on I, then*

(1) *If $f'' > 0$ on I, then f is* ***convex (concave up)*** *on I.*
(2) *If $f'' < 0$ on I, then f is* ***concave (concave down)*** *on I.*

Example. Determine the convexity of $y = 3 + \sin(x)$ on $[0, 2\pi]$.

Solution. First, note that we have a trigonometric function, which is infinitely differentiable. Hence, we can use the second derivative test for convexity/concavity. We compute the 2nd derivative and find $y''(x) = -\sin(x)$.

Now, we ask, for which $x \in [0, 2\pi]$ is $y'' > 0$ and $y'' < 0$ respectively (recall that we did something similar to y').

$$y'' = -\sin x > 0 \implies \sin(x) < 0 \implies \pi < x < 2\pi$$
$$y'' = -\sin x < 0 \implies \sin(x) > 0 \implies 0 < x < 2\pi$$

Therefore, $y = 3 + \sin(x)$ is convex/concave up on $(\pi, 2\pi)$ and concave/concave down on $(0, \pi)$.

Note that the point $(\pi, 3)$ is where the second derivative vanishes. The convexity of y changes from concave (concave down) to convex (concave up). Recall that in the First Derivative Test, we studied the sign of the First Derivative near critical points and used a similar "argument". The point $(\pi, 3)$ here simply signifies that convexity of the function changes. It's special enough to warrant the following definition.

Definition. A point $(c, f(c))$ where the graph of a function has a tangent line and the convexity changes is a **point of inflection** (POI).

Remark. Note that this definition only depends on $f'(c)$ existing, and assume nothing about f''. A point of inflection is a notion in parallel to **local extrema**, which does NOT depend on the notion of a derivative.

Proposition. *[A necessary condition for POI] If f is twice differentiable, then at the point of inflection, we have $f'' = 0$ or undefined.*

Remark. Once again, this implication is only one way. The converse in general is not true. So even if you checked that certain point satisfies $f''(c) = 0$, it's not **necessarily** a point of inflection. You must study the convexity near this point (just like you study increasing/decreasing near critical points).

Example. [counterexample to the converse of the proposition: if $f''(c) = 0$ then $x = c$ is a POI (false statement)].

Consider $f(x) = x^4$. Certainly $f''(0) = 0$. However, $f''(x) = 12x^2$, which means the convexity is **always** positive, especially near $x = 0$. There is NO convexity change through $x = 0$. Thus $x = 0$ is NOT a point of inflection.

Example. Consider $f(x) = x^{5/3}$. We find

$$f''(x) = \frac{10}{9} x^{-\frac{1}{3}},$$

which is undefined at $x = 0$. This makes $x = 0$ a **potential** POI, and we confirm it by looking at the sign of the second derivative near it.

Indeed, for $x < 0$, $f''(x) < 0$, and for $x > 0$, $f''(x) > 0$. So it is concave for $x < 0$ and convex for $x > 0$. This qualifies $x = 0$ as a POI.

Theorem. *[Second derivative test for local extrema]*

Suppose f'' is continuous on an open interval containing $x = c$.

(1) *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.*
(2) *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.*
(3) *If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, minimum or neither.*

Curve sketching is the **most comprehensive process** that checks your understanding and technical skills in Calculus I. Sketching curves properly with precision and correctness is a strong indicator of your understanding of Calculus I materials.

We take the Second Derivative Test directly to practice with a detailed curve sketching example

$$f(x) = \frac{x}{1+x^2}$$

by following the steps:

(1) [Basic study]
   (a) Identify the domain of f and symmetries (odd or even) the curve may have.
   (b) Identify any vertical and horizontal asymptotes that may exist.
   (c) Identify key points, such as the x- and y-intercepts.
(2) [1st derivative study] Find the first derivative.
   (a) Find the critical points.
   (b) Find where the curve is increasing and where it is decreasing.
(3) [2nd derivative study] Find the second derivative
   (a) Find the values such that $f'' = 0.$, i.e., the candidate POIs.
   (b) Determine the convexity of the curve by studying the sign of f'' in between the candidate POIs.
(4) Sketch the curve.

MATH 1021 LECTURE XXII
APPLIED OPTIMIZATION

(1) Read the problem statements **very carefully**.
(2) Identify the known quantity and relationship (known as the constraints).
(3) Introduce all variables. If applicable, draw a figure and label all variables. Write down their possible range of values.
(4) Identify what you need to optimize (in physical terms, such as area, volume, total cost, etc.). Sometimes this is given clearly. Sometimes, you need to interpret.
(5) Express what you need to optimize in the simplest manner possible, which may involve multiple variables.
(6) Shrink the number of variables by using the known quantity and relationship.
(7) Do calculus to find absolute optima.

Constrained optimization problems occur practically everyday.

What's the optimal amount of muffins one should bake to maximize profit while the cost is subject to budget constraints.

What's the "perfect" amount of hours to study for Midterm II so that you learn the most while not burning out due to having a fixed amount of energy?

Example. Design a one-liter (1000 cubic centimeters) can shaped like a right circular cylinder. What dimensions will use the least material?

Solution. Volume of the cylinder can be computed when we know the radius of the base circle r and the height of the cylinder h. In this problem, they are related by

$$\pi r^2 h = 1000.$$

This is known as **the constraint** because we require that the can is designed to hold this amount of fluid.

At the same time, the material one uses pertains to the total surface area of the cylinder, that is,

$$S = 2\pi r h + 2\pi r^2,$$

which we aim to minimize, given the constraint.

Altogether, the problem can written in the standard form of a constrained optimization problem.

$$\min_{r,h>0} S(r,h) = 2\pi r h + 2\pi r^2$$
$$\text{subject to} \pi r^2 h = 1000.$$

So domain of the function, for practical uses, is $(r,h) \in (0,\infty) \times (0,\infty)$, a subset of $\mathbb{R}^2$ – they should be positive numbers.

Luckily, using the constraint, we can express h as a function of r, meaning that we can reduce the number of independent variables from 2 to 1.

$$h = \frac{1000}{\pi r^2}.$$

Therefore, the minimization becomes

$$\min_{r>0} S(r) = 2\pi r \left(\frac{1000}{\pi r^2}\right) + 2\pi r^2 = \frac{2000}{r} + 2\pi r^2.$$

To find the minimum of this function, we first find the critical points. They satisfy

$$0 = S'(r) = -\frac{2000}{r^2} + 4\pi r \implies r = \left(\frac{500}{4}\right)^{\frac{1}{3}}$$

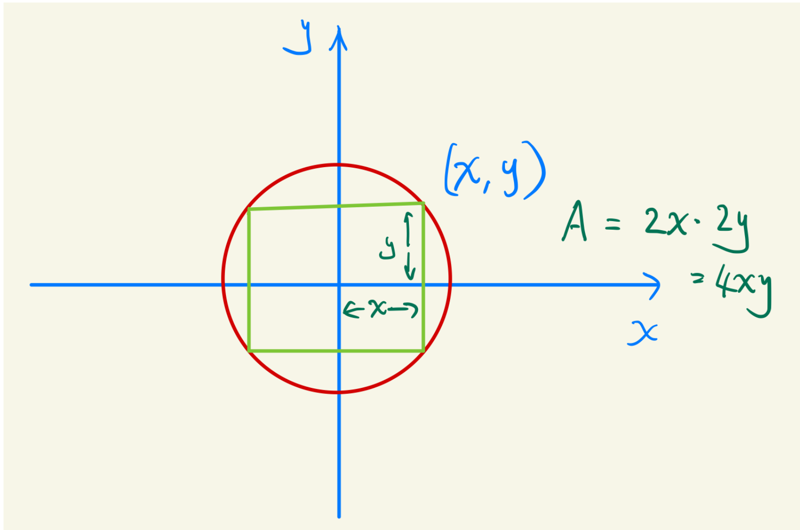
and also $r = 0$ is a critical point since it makes S' undefined. But $r = 0$ is not a feasible solution since a cylinder of radius 0 holds nothing. We reject $r = 0$.

So, is $r = \left(\frac{500}{4}\right)^{\frac{1}{3}}$ guaranteed to be a minimum? No, you need to verify it, using the second derivative test, i.e.,

$$S''(r) = \frac{4000}{r^3} + 4\pi > 0, \quad \text{for } r > 0.$$

Thus, any critical point you find is a local minimum. This is a global minimum because there is only **one** critical point.

Example. A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?



Solution. The area of a rectangle is given by

$$A(x,y) = 4xy$$

where x is the half length and y is the half width (see sketch). So, $x > 0$, and $y > 0$.

If there is no constraint, then just increase x and y without bounds to form the "largest" area. Here, the point (x,y) lives on the circle in the first quadrant, satisfying the equation

$$x^2 + y^2 = 2^2 \implies y = \sqrt{4 - x^2}.$$

Thus, the area of a rectangle with its vertex constrained to this circle is then

$$A(x) = 4x\sqrt{4 - x^2}, \quad 0 \le x \le 2,$$

where we used the relationship between x and y.

To maximize $A(x)$, we find its critical point that satisfy

$$0 = A'(x) = 4\sqrt{4 - x^2} - \frac{4x^2}{\sqrt{4 - x^2}} = \frac{4\left(4 - x^2\right) - 4x^2}{\sqrt{4 - x^2}} = \frac{16 - 8x^2}{\sqrt{4 - x^2}}.$$

Thus,

$$16 - 8x^2 = 0 \implies x = \pm\sqrt{2}.$$

We reject $x = -\sqrt{2}$ since we have already required that $x > 0$.

Note that this function also has endpoints $x = 0$ and $x = 2$. We find that

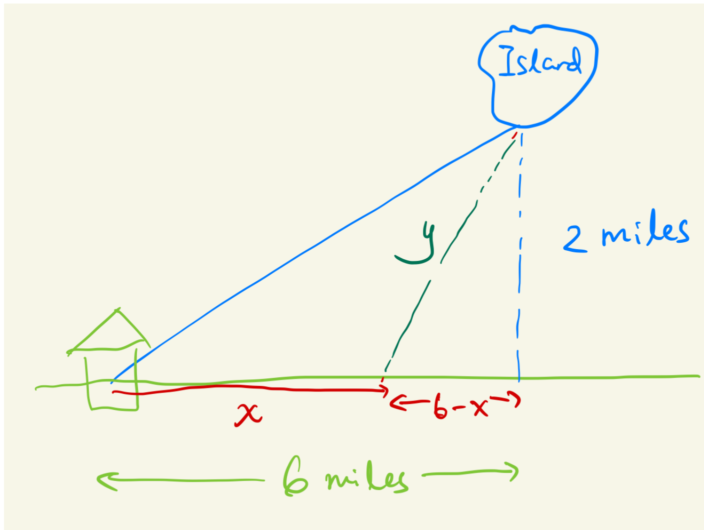
$$A(0) = 0$$
$$A(2) = 0$$
$$A\left(\sqrt{2}\right) = 4\sqrt{2}\sqrt{4 - \left(\sqrt{2}\right)^2} = 8.$$

By the Theorem [Absolute Extrema for Differentiable Functions; see lecture 20], a differentiable function on $[a,b]$ will achieve its absolute max and min either at the endpoints or at critical points. Here, very clearly, $\left(\sqrt{2}, 4\right)$ is the absolute maximum of $A(x)$. The corresponding height

$$y = \sqrt{4 - x^2} = \sqrt{4 - \left(\sqrt{2}\right)^2} = \sqrt{2}.$$

This means, the largest rectangle that can be inscribed inside a circle of radius 2 is a circle of side length $2\sqrt{2}$. Geometrically, it also makes sense because the radius 2 is the half length of the diagonal of the square – the side length then is $\frac{2\times 2}{\sqrt{2}} = 2\sqrt{2}$.

Example. An island is 2 miles due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore that is 6 miles west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of 8 miles per hour and swims at a rate of 3 miles. How far should the visitor run before swimming to minimize the time it takes to reach the island?



Solution. We want to minimize total travel time T. This total time consists of running time T_{run} and swimming time T_{swim}, via

$$T = T_{\text{run}} + T_{\text{swim}}.$$

Suppose the visitor runs x miles and swims y miles. Then, his travel time is

$$T(x, y) = \frac{x}{8} + \frac{y}{3}, \quad 0 \le x \le 6, \quad y > 0.$$

This is a two dimensional problem, which we don't know how to solve. What's more, we haven't made use of the special geometry between the island and the cabin.

Denote the closest point along the shore to the island D. Then, the location the visitor decides to enter water is $(6 - x)$ miles from D. Thus, we have a right triangle with side lengths $(6 - x)$ and 2 and hypotenuse y.

$$y^2 = (6 - x)^2 + 2^2 \implies y = \sqrt{(6 - x)^2 + 4}$$

(note that we don't take the negative part of the square root because $y > 0$).

Plugging this expression into $T(x, y)$, we have eliminated one independent variable

$$T(x) = \frac{x}{8} + \frac{\sqrt{(6 - x)^2 + 4}}{3}, \quad 0 \le x \le 6.$$

To minimize, we find the critical points of T that satisfy

$$0 = T'(x) = \frac{1}{8} + \frac{1}{3} \frac{2(6 - x)(-1)}{2\sqrt{(6 - x)^2 + 4}} = \frac{1}{8} - \frac{6 - x}{3\sqrt{(6 - x)^2 + 4}}.$$

Now, isolating the variables, we have

$$\frac{1}{8} = \frac{6-x}{3\sqrt{(6-x)^2+4}}$$

$$\overset{\text{cross multiply}}{\Longrightarrow} 3\sqrt{(6-x)^2+4} = 8(6-x)$$

$$\overset{\text{square both sides}}{\Longrightarrow} 9\left[(6-x)^2+4\right] = 64(6-x)^2$$

$$\Longrightarrow 9(6-x)^2+36 = 64(6-x)^2$$

$$\Longrightarrow 36 = 55(6-x)^2$$

$$\Longrightarrow 6-x = \pm\frac{6}{\sqrt{55}}$$

$$\Longrightarrow x = 6 \pm \frac{6}{\sqrt{55}}$$

We reject $x = 6 + \frac{6}{\sqrt{55}} > 6$. Write $x^* = 6 - \frac{6}{\sqrt{55}}$ as a shorthand.

Now, we check the endpoints,

$$T(0) = \frac{\sqrt{40}}{3} = \frac{4}{3}\sqrt{10} = \frac{16\sqrt{10}}{12}.$$

$$T(6) = \frac{3}{4} + \frac{2}{3} = \frac{17}{12} = \frac{3}{4} + \frac{8}{12}.$$

$$T(x^*) = \frac{3}{4} - \frac{3}{4\sqrt{55}} + \frac{16}{3\sqrt{55}} = \frac{3}{4} + \frac{\sqrt{55}}{12}.$$

Without a calculator, we can still rewrite the fractions to judge which number is bigger. First, $T(0) = \frac{16\sqrt{10}}{12} > \frac{17}{12} = T(6)$. Then, $T(6) = \frac{3}{4} + \frac{8}{12} > \frac{3}{4} + \frac{\sqrt{55}}{12} = T(x^*)$.

Clearly,

$$T(0) > T(6) > T(x^*),$$

which means $T(x^*)$ is the absolute minimum.

MATH 1021 LECTURE XXIII
THE MEAN VALUE THEOREM AND ANALYTICAL ROOT-FINDING

1. The Mean Value Theorem

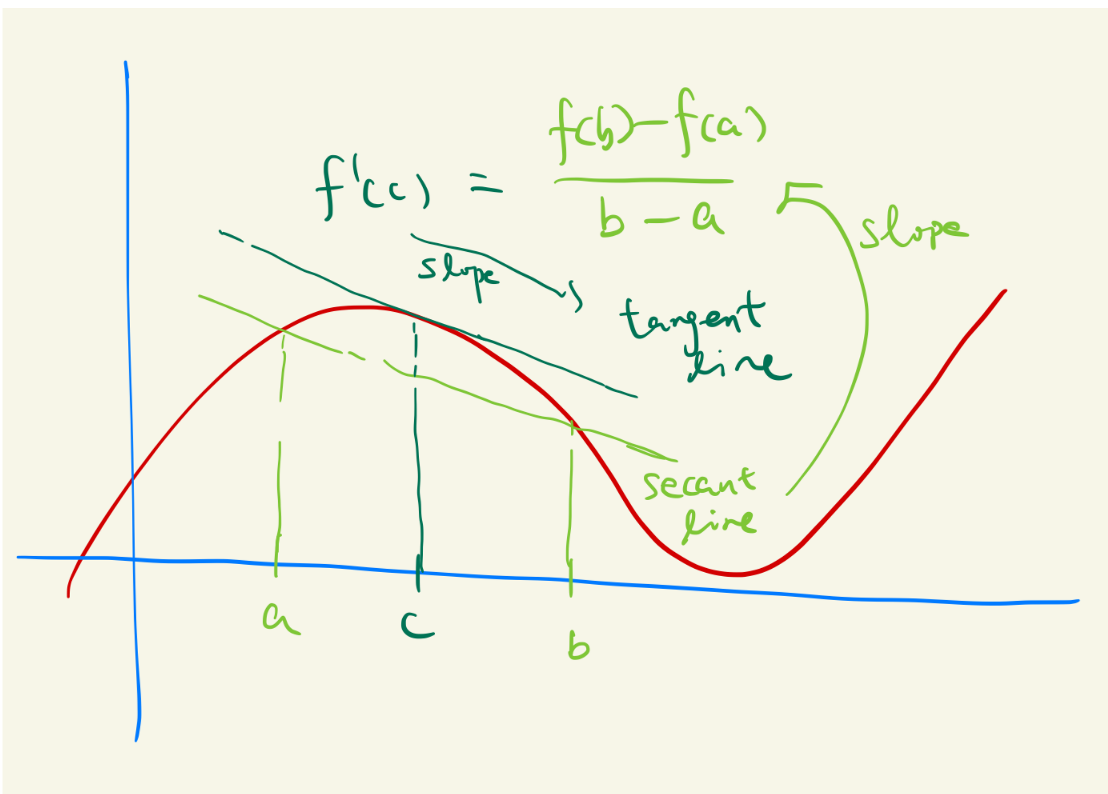
From the Intermediate Value Theorem, we've learned that a continuous function achieves every value between $f(a)$ and $f(b)$ on $[a,b]$. So, what can we say about a differentiable function and its slope value on the same interval? The observation is a geometric one. This result helps us justify why $f' > 0$ implies that the function is increasing (though it seems obvious).

Theorem. *[Mean Value Theorem] If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists a number $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

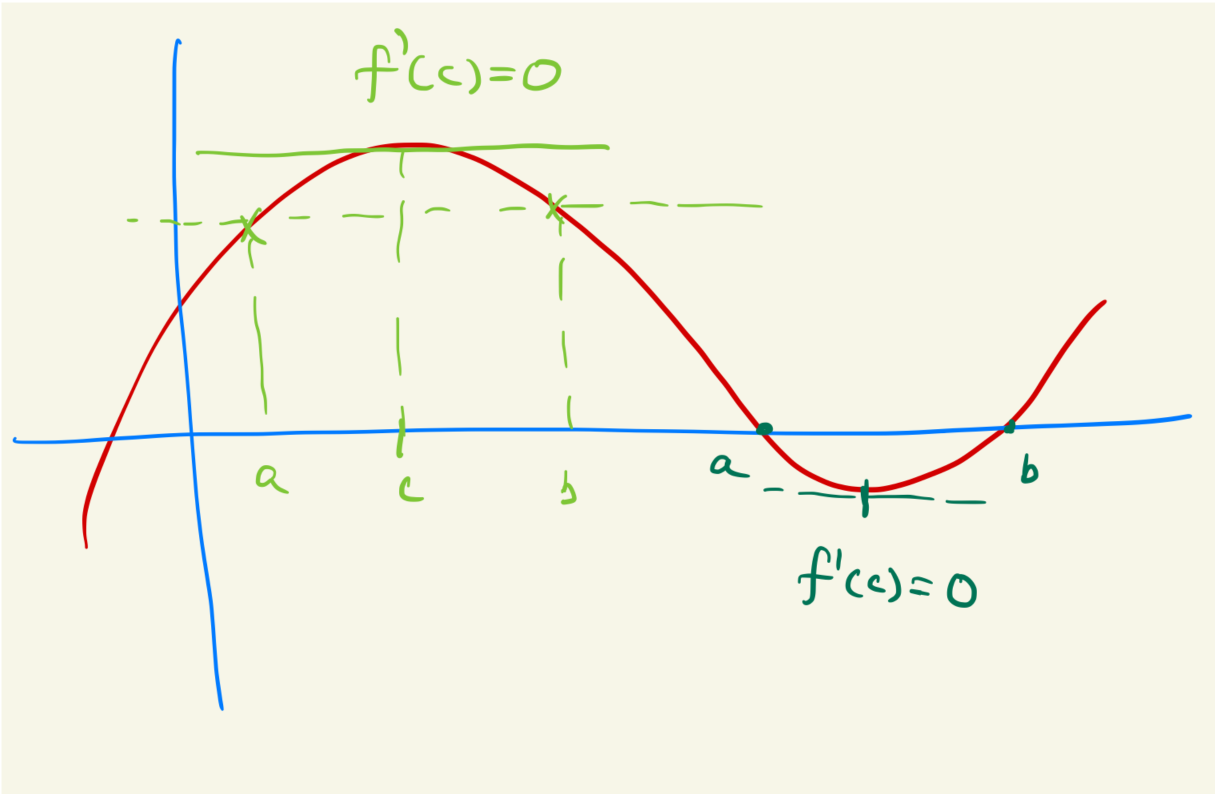
In other words, somewhere on this interval, the function f achieves a slope equal to that of the secant line between $(a, f(a))$ and $(b, f(b))$.

One picture speaks more loudly than a proof.



Corollary. *[Rolle's Theorem] With the same conditions as MVT, if $f(a) = f(b)$, then there exists at least a $c \in (a,b)$ such that $f'(c) = 0$.*

One picture speaks more loudly than a proof.



Remark. Rolle's theorem is often used in a proof technique called "proof by contradiction". The procedure is as follows. Suppose you want to show that statement A is true. You always keep track of a pool of truths already given or obtained. First you assume that statement A is false. This could possibly imply that one of the statements in the pool of truth is false. Now, as a statement cannot be both true and false at the same time, this can only mean that your initial assumption that statement A is false is wrong. This implies, in turn, that A must be true.

2. Application: Analytical Root-finding

We can show precisely whether an equation has exactly a solution, either to pinpoint where it is, or to guide a numerical method.

Example. Show that there is exactly one real solution to the equation $x^3 + 3x + 1 = 0$.
Define $f(x) = x^3 + 3x + 1$.

(1) Show there is **at least** one real zero of f via the Intermediate Value Theorem.
(2) Show there is **at most** one real zero of f via Rolle's Theorem and proof by contradiction.
(3) Conclude with step 1 and 2 that there must be **exactly** one real solution.

We explain each step more carefully.

2.1. At least one real solution via IVT.

If a continuous function changes sign at the endpoints of a closed interval. Here, we find

$$f(0) = 1 > 0,$$
$$f(-1) = -1 - 3 + 1 = -3 < 0.$$

Therefore, there is **at least** one such $c \in (-1, 0)$ such that $f(c) = 0$.

2.2. **At most one real solution via Rolle's.** It's difficult to show **at most one real solution** directly. However, let's argue otherwise. **Suppose the contrary** that there are **more than two zeros of** f on $(-1,0)$. More precisely, let $x_1 < x_2$ be two points in this interval such that $f(x_1) = 0$ and $f(x_2) = 0$. Then, by Rolle's theorem, there exists $d \in (x_1, x_2)$ such that

$$f'(d) = 0.$$

However, $f'(x) = 3x^2 + 3 > 0$ for all $x \in (-1,0)$, i.e., f' is never zero on this interval. This contradicts the conclusion of Rolle's theorem, which means, our assumption that "there are **two zeros** of f on $(-1,0)$" is wrong. Reductio ad absurdum! Therefore, there can **at most be one real zero of** f.

2.3. **Conclude: Sections 2.1+2.2 means we have precisely one real zero of f.**

Remark. Upon further observation, we find that f is monotone increasing on $(-1,0)$. Indeed, monotone functions can only cross the x-axis at most one times. Here, we are lucky to have found that $f' > 0$ not only on $(-1,0)$, but also for every x in its domain.

3. Numerical Root-finding: Newton-Raphson Method

The Intermediate Value Theorem only tells us where a root possibly lies, and there may be more than one of them there. Now, with the improvement of Rolle's theorem, we can really hone in on an interval and claim that there is only one root there. Yet with all these Theorem, they only provide **existence** of a root, not where it is. It still requires some numerical methods to find precisely where the roots are. If you can't think of an application of root-finding, just recall that to optimize an objective function, one needs to find the critical point, which is precisely the rootfinding problem for $f'(x)$.

So, how can we do better with numerics? Is there an algorithm which we can use to find the roots, guided by an initial pen-and-paper analysis?

The knowledge of linearization comes back. For every differentiable function, we can approximate it locally by its tangent line, given by

$$L_{x=x_0}(x) = f(x_0) + f'(x_0)(x - x_0)$$

where x_0 is the point of linearization/center. We continue with the fact that $L_{x=x_0}(x)$ is an approximant of f, in every aspect – a dream, but a worthy one. One such aspect is that the zero of $L_{x=x_0}$ should be somewhat close to that of f. The zero of $L_{x=x_0}$, labeled as $x = x_1$,

$$L_{x=x_0}(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) = 0 \implies x_0 - x_1 = \frac{f(x_0)}{f'(x_0)} \implies x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Certainly, one can check if $f(x_1) = 0$; if so, we are done. However, if not, we are going to make a second linearization

$$L_{x=x_1}(x) = f(x_1) + f'(x_1)(x - x_1)$$

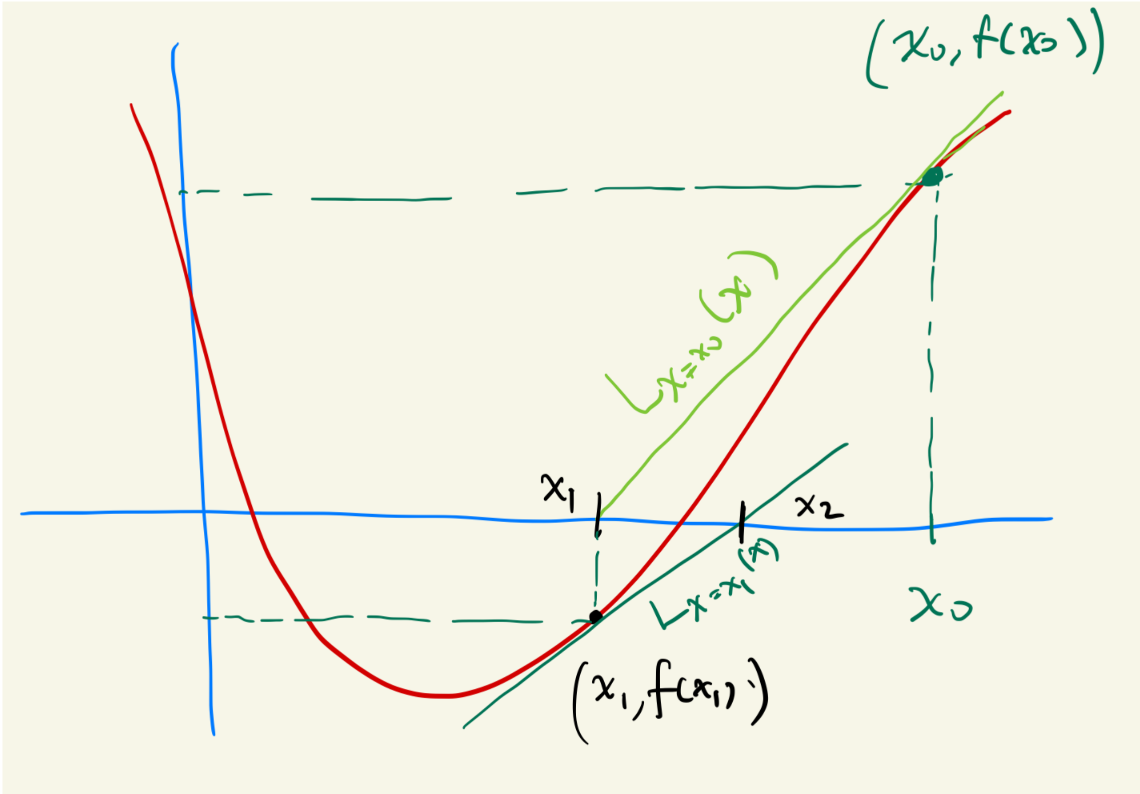
which, by the same logic, has a zero $x = x_2$ at

$$L_{x=x_1}(x_2) = 0 \implies x_2 = x_1 - \frac{f(x_0)}{f'(x_0)}.$$

Check if $f(x_2) = 0$ again. If not, continue this **iterative** procedure,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Here, we note that if $f(x_n) = 0$, that is, at the n^{th} step, we find the root, then $x_{n+1} = x_n$, which means this continued search is halted. We have successfully found the root of f via **iteration**. This method is called the Newton-Raphson method.



$(x_0, f(x_0))$
$L_{x=x_0}(x)$
x_1
x_2
$L_{x=x_1}(x)$
x_0
$(x_1, f(x_1))$