RECITATION IV

NAME: SOLUTIONS

Problem 1. Compute $\int \frac{3x^2+2}{(x^2+3)(x-2)}dx$

Solution. We identify an irreducible quadratic x^2+3. Partial fractions are of the form

$$\begin{aligned}\frac{3x^2+2}{(x^2+3)(x-2)} &= \frac{Ax+B}{x^2+3}+\frac{C}{x-2} \\ &= \frac{(Ax+B)(x-2)+C\left(x^2+3\right)}{(x^2+3)(x-2)} \\ &= \frac{Ax^2+Bx-2Ax-2B+Cx^2+3C}{(x^2+3)(x-2)} \\ &= \frac{(A+C)x^2+(B-2A)x+(3C-2B)}{(x^2+3)(x-2)}\end{aligned}$$

which yields the system

$$\begin{aligned}A+C&=3\\ B-2A&=0\\ 3C-2B&=2\end{aligned}$$

This has solution

$$\begin{aligned}A&=1\\ B&=2\\ C&=2\end{aligned}$$

hence

$$\frac{3x^2+2}{(x^2+3)(x-2)}=\frac{x+2}{x^2+3}+\frac{2}{x-2}$$

The integral can be found by splitting,

$$\begin{aligned}\int \frac{3x^2+2}{(x^2+3)(x-2)}dx &= \int\left(\frac{x+2}{x^2+3}+\frac{2}{x-2}\right)dx \\ &= \int \frac{x}{x^2+3}dx+2\int\frac{1}{x^2+3}dx+2\int\frac{1}{x-2}dx \\ &= \frac{1}{2}\ln\left|x^2+3\right|+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+2\ln|x-2|+C\end{aligned}$$

where the second integral comes from the $\int \frac{1}{x^2+a^2}dx$-type where $a=\sqrt{3}$.

Problem 2. Estimate the integral by Trapezoidal rule with 4 subintervals: $\int_1^3 \frac{3}{x-(1/2)}dx$. Determined the error by comparing to the exact solution.

Solution. Determine the following things:

$$\begin{aligned}\Delta x &= \frac{b-a}{n}=\frac{3-1}{4}=\frac{1}{2}\\ x_k &= \left\{1,\frac{3}{2},2,\frac{5}{2},3\right\},\quad k=0,\ldots,4\\ f(x) &= \frac{3}{x-1/2}\end{aligned}$$

Then

$$\begin{aligned}
T_4 &= \frac{\Delta x}{2}\left(f\left(x_0\right)+2f\left(x_1\right)+2f\left(x_2\right)+2f\left(x_3\right)+f\left(x_4\right)\right)\\
&= \frac{1}{4}\left(\frac{3}{1-1/2}+2\cdot\frac{3}{3/2-1/2}+2\cdot\frac{3}{2-1/2}+2\cdot\frac{3}{5/2-1/2}+\frac{3}{3-1/2}\right)\\
&= \frac{1}{4}\left(6+6+4+3+\frac{6}{5}\right)\\
&= \frac{1}{4}\left(\frac{95+6}{5}\right)\\
&= \frac{101}{20}\\
&= 5.05\,(\text{only for comparison purposes})
\end{aligned}$$

Exact answer is

$$\int_1^3 \frac{3}{x-(1/2)}dx = 3\ln|x-1/2|\,|_{x=1}^{x=3} = 3\left(\ln\frac{5}{2}-\ln\frac{1}{2}\right) = 3\ln 5 \approx 4.8283$$

It is an overestimate. (Can you tell why?)

Problem 3. Which of the following integrals are improper? If so, of which type, and how would you proceed to find its value?

(1) $\int_3^\infty \frac{3}{(x-2)^2}dx$

Solution. Yes, of the 1st type as it involves infinities.

$$\begin{aligned}
\int_3^\infty \frac{3}{(x-2)^2}dx &= 3\lim_{b\to\infty}\int_3^b \frac{1}{(x-2)^2}dx\\
&= -3\lim_{b\to\infty}\frac{1}{x-2}\,|_3^b\\
&= -3\left(\frac{1}{b-2}-1\right)\\
&= 3
\end{aligned}$$

(2) $\int_0^2 \frac{3}{(x-2)^2}dx$

Solution. Yes, of the 2nd type case (a) as it involves the upper limit hitting a discontinuity (vertical asymptote at $x=2$)

$$\begin{aligned}
\int_0^2 \frac{3}{(x-2)^2}dx &= 3\lim_{a\to 2^-}\int_0^a \frac{1}{(x-2)^2}dx\\
&= -3\lim_{a\to 2^-}\frac{1}{x-2}\,|_0^a\\
&= -3\lim_{a\to 2^-}\left(\frac{1}{a-2}+\frac{1}{2}\right)\\
&= +\infty
\end{aligned}$$

(3) $\int_1^3 \frac{3}{(x-2)^2}dx$

Solution. Yes, of the 2nd type case (c) as it jumps over a discontinuity (vertical asymptote at $x = 2$)

$$\begin{aligned}
\int_1^3 \frac{3}{(x-2)^2} dx &= \int_1^2 \frac{3}{(x-2)^2} dx + \int_2^3 \frac{3}{(x-2)^2} dx \\
&= 3 \lim_{a \to 2^-} \int_1^a \frac{1}{(x-2)^2} dx + 3 \lim_{b \to 2^+} \int_b^3 \frac{1}{(x-2)^2} dx \\
&= -3 \left[\lim_{a \to 2^-} \frac{1}{x-2} |_1^a + \lim_{b \to 2^+} \frac{1}{x-2} |_b^3 \right] \\
&= -3 \left[\lim_{a \to 2^-} \left(\frac{1}{a-2} + \frac{1}{2} \right) + \lim_{b \to 2^+} \left(1 - \frac{1}{b-2} \right) \right] \\
&= -3 \left[-\infty + \frac{3}{2} + -\infty \right] \\
&= +\infty
\end{aligned}$$

(Why the negative infinities in the second last line?)

Problem 4. Find the partial fractions to $\frac{3x}{(x-4)(x+1)^2}$. See if you can integrate $\int \frac{3x dx}{(x-4)(x+1)^2}$.

Solution. We identify a repeated factor $(x+1)$ and thus must include powers up to its multiplicity, 2 in this case. Partial fractions are of the form

$$\begin{aligned}
\frac{3x}{(x-4)(x+1)^2} &= \frac{A}{x-4} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \\
&= \frac{A(x+1)^2 + B(x-4)(x+1) + C(x-4)}{(x-4)(x+1)^2} \\
&= \frac{Ax^2 + 2Ax + A + Bx^2 - 3Bx - 4B + Cx - 4C}{(x-4)(x+1)^2} \\
&= \frac{(A+B)x^2 + (2A - 3B + C)x + (A - 4B - 4C)}{(x-4)(x+1)^2}
\end{aligned}$$

which yields the system

$$\begin{aligned}
A + B &= 0 \\
2A - 3B + C &= 3 \\
A - 4B - 4C &= 0
\end{aligned}$$

This has solution

$$\begin{aligned}
A &= \frac{12}{25} \\
B &= -\frac{12}{25} \\
C &= \frac{3}{5}
\end{aligned}$$

and thus altogether partial fractions

$$\frac{3x}{(x-4)(x+1)^2} = \frac{12}{25} \left(\frac{1}{x-4} - \frac{1}{x+1} \right) + \frac{3}{5(x+1)^2}$$

The integral is straightforward,

$$\begin{aligned}
\int \frac{3x dx}{(x-4)(x+1)^2} &= \frac{12}{25} \int \left(\frac{1}{x-4} - \frac{1}{x+1} \right) dx + \frac{3}{5} \int \frac{1}{(x+1)^2} dx \\
&= \frac{12}{25} \left[\ln|x-4| - \ln|x+1| \right] - \frac{3}{5(x+1)} + C \\
&= \frac{12}{25} \ln \left| \frac{x-4}{x+1} \right| - \frac{3}{5(x+1)} + C
\end{aligned}$$