RECITATION IV

NAME:

Problem 1. Find the volume of the shape generated by revolving the curve $y = \sqrt{3-2x}$, $-6 \leq x \leq 0$ about the x-axis.

Solution. Volume:
Method: Disk.
Reason: axis of revolution is contained in the region.

$$V = \int_{-6}^{0} \pi \left(\sqrt{3-2x}\right)^2 dx = \pi \int_{-6}^{0} (3-2x)\, dx = \pi \left[3x - x^2\right]_{x=-6}^{x=0}$$
$$= \pi \left[0 - 3(-6) + (-6)^2\right] = 54\pi$$

Area of surface of revolution:
Radius: $f(x) = \sqrt{3-2x}$
Arc length element: $ds = \sqrt{1 + (f'(x))^2} dx = \sqrt{1 + \left(\frac{-1}{\sqrt{3-2x}}\right)^2} dx = \sqrt{\frac{4-2x}{3-2x}} dx$

$$\begin{aligned}
A &= 2\pi \int_{-6}^{0} f(x) \sqrt{1 + (f'(x))^2} dx \\
&= 2\pi \int_{-6}^{0} \sqrt{3-2x} \sqrt{\frac{4-2x}{3-2x}} dx \\
&= 2\pi \int_{-6}^{0} (4-2x)^{\frac{1}{2}}\, dx \\
&= -\frac{2}{3}\pi (4-2x)^{\frac{3}{2}} \,|_{-6}^{0} \\
&= -\frac{2}{3}\pi \left[2^{\frac{3}{2}} - 16^{\frac{3}{2}}\right] \\
&= \frac{2}{3}\pi \left[64 - 2\sqrt{2}\right]
\end{aligned}$$

Problem 2. Evaluate $\int_{\ln 2}^{\ln 3} \sinh(2x)\, dx$

Solution. Straightforward. Must remember definition of $\cosh(x)$ and replace x by $2x$.

$$\begin{aligned}
I &= \frac{\cosh(2x)}{2} \,|_{\ln 2}^{\ln 3} \\
&= \frac{e^{2x} + e^{-2x}}{4} \,|_{\ln 2}^{\ln 3} \\
&= \frac{1}{4}\left(e^{2\ln 3} + e^{-2\ln 3} - e^{2\ln 2} - e^{-2\ln 2}\right) \\
&= \frac{1}{4}\left(9 + \frac{1}{9} - 4 - \frac{1}{4}\right) \\
&= \frac{1}{4}\left(5 - \frac{5}{36}\right) \\
&= \frac{175}{144}
\end{aligned}$$

Problem 3. A solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis between these planes are squares whose bases run from the parabola $y = 2 - x^2$ to $y = x^2 + 5$. Find the volume of the solid.

Solution. Draw the shape. Square cross-sections $A = s^2$ where s is the length of each size, which runs from the lower parabola $y = 2 - x^2$ to the upper parabola $y = x^2 + 5$. Hence, $s = x^2 + 5 - \left(2 - x^2\right) = 2x^2 + 3$and thus $A\left(x\right) = \left(2x^2 + 3\right)^2$. Altogether,

$$V = \int_{-1}^{1} A\left(x\right) dx = \int_{-1}^{1} \left(2x^2 + 3\right)^2 dx = \int_{-1}^{1} \left(4x^4 + 12x^2 + 9\right) dx = 2\int_{0}^{1} \left(4x^4 + 12x^2 + 9\right) dx = 2\left[\frac{4}{5} + 4 + 9\right] = \frac{138}{5}$$

Problem 4. The graph of $y = x^{2/3}$, $0 \le x \le 8$ is revolved about the y-axis to form a tank. If the tank is filled with a very light liquid weighing 30 lb/ft^3, find the work required to pump the liquid to a level 2 ft above the top of the tank.

Solution. Draw the graph. Form the flat slab, with cross-section perpendicular to y-axis. Integrate in y. Note, the tank tops at $y\left(8\right) = 8^{2/3} = 4$. So the target location is the line $y = 4 + 2 = 6$ ft. The range of the liquid in y is $0 \le y \le 4$.

$$\begin{aligned} \Delta V &= \pi r^2\left(y\right)\Delta y = \pi\left(y^{\frac{3}{2}}\right)^2 \Delta y \\ F_{slab} &= \rho \Delta V = \rho\pi y^3 \Delta y \\ \Delta W &= F_{slab} \times \text{distance required to lift this slab} \\ &= \rho\pi y^3 \Delta y \times \left(6 - y\right) \end{aligned}$$

$$\begin{aligned} W = \int dW &= \int_0^4 \rho\pi y^3\left(6 - y\right) dy = \rho\pi\left[\frac{3}{2}y^4 - \frac{y^5}{5}\right]_{y=0}^{y=4} \\ &= \rho\pi\left(384 - \frac{1024}{5}\right) = \rho\pi\frac{896}{5} = 5376\pi \end{aligned}$$

Problem 5. Find the volume of the solid generated by revolving the region in the first quadrant bounded by $y = \sqrt{x}$, $y = 2$ and $x = 0$ about the line $x = -1$ **by the shell method**.

Solution. Sketch the region. It is the region bounded by $y = 2$, $x = 0$ and above the square root curve. Domain is $0 \le x \le 4$ since the curve on the right must high $y = 2$ (by setting $x = 4$).

$$\begin{aligned} V &= 2\pi\int_0^4 \text{radius} \times \text{height} dx \\ &= 2\pi\int_0^4 \left(x + 1\right)\left(2 - \sqrt{x}\right) dx \\ &= 2\pi\int_0^4 \left(2x + 2 - x^{\frac{3}{2}} - x^{\frac{1}{2}}\right) dx \\ &= 2\pi\left[x^2 + 2x - \frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}}\right]_{x=0}^{x=4} \\ &= 2\pi\left[16 + 8 - \frac{2}{5}\left(4\right)^{\frac{5}{2}} - \frac{2}{3}\left(4\right)^{\frac{3}{2}}\right] \\ &= 2\pi\left[24 - \frac{2}{5} \times 32 - \frac{2}{3} \times 8\right] \\ &= 2\pi\left[24 - \frac{64}{5} - \frac{16}{3}\right] \\ &= 2\pi\left[\frac{360 - 192 - 80}{15}\right] \\ &= \frac{176}{15}\pi \end{aligned}$$