MATH 111-007 RECITATION 1201

Problem 1. Write the sums without sigma notation and then evaluate them.

(1)

$$\sum_{k=1}^{3} (-1)^{k+1} \sin\left(\frac{\pi}{k}\right)$$

Solution.

$$\begin{aligned}\sum_{k=1}^{3} (-1)^{k+1} \sin\left(\frac{\pi}{k}\right) &= (-1)^{1+1} \sin\left(\frac{\pi}{1}\right) + (-1)^{1+2} \sin\left(\frac{\pi}{2}\right) + (-1)^{1+3} \sin\left(\frac{\pi}{3}\right) \\ &= \sin(\pi) - \sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{3}\right) \\ &= 0 - 1 + \frac{\sqrt{3}}{2} \\ &= \frac{\sqrt{3}}{2} - 1\end{aligned}$$

(2)

$$\sum_{l=1}^{4} (-1)^{l} \cos(l\pi)$$

Solution.

$$\begin{aligned}\sum_{l=1}^{4} (-1)^{l} \cos(l\pi) &= (-1)^{1} \cos(\pi) + (-1)^{2} \cos(2\pi) + (-1)^{3} \cos(3\pi) + (-1)^{4} \cos(4\pi) \\ &= -\cos(\pi) + \cos(2\pi) - \cos(3\pi) + \cos(4\pi) \\ &= -(-1) + 1 - (-1) + 1 \\ &= 4\end{aligned}$$

Problem 2. Express the following sums in sigma notation.

(1)

$$1 + 4 + 9 + 16$$

Solution.

$$1 + 4 + 9 + 16 = \sum_{k=1}^{4} k^2$$

or you can write in many other ways,

$$1 + 4 + 9 + 16 = \sum_{k=0}^{3} (k+1)^2 = \sum_{k=2}^{5} (k-1)^2 = \cdots$$

Note the slightest change here? Take $\sum_{k=1}^{4} k^2$ as your reference case here. Then, if you want to shift the index down (also known as to the left) by 1, say, to $\sum_{k=0}^{3}$, then you must also shift k to the left by doing $k+1$ (remember translating a function $f(x)$ one unit to the left is to do $f(x+1)$?).

(2)

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$$

Solution. Whenever you see alternating signs, you immediately think of powers of -1, i.e. $(-1)^k$. Now, how the signs alternate depend on the sign of the first term. Here, it is a positive sign. So,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}=\sum_{k=1}^{5}(-1)^{k+1}\frac{1}{k}.$$

This is not the only way. The reason why we use $(-1)^{k+1}$ is because our index k started at $k=1$, so that the first term has a positive sign as required. You may start the index at $k=0$, to obtain

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}=\sum_{k=0}^{4}(-1)^{k}\frac{1}{k+1},$$

because we shifted the index one unit down (or one unit to the left).

(3)
$$-\frac{1}{5}+\frac{2}{5}-\frac{3}{5}+\frac{4}{5}-\frac{5}{5}$$

Solution. Alluding the method discussed in part (2), we notice that the first term has a negative sign. Therefore, if you choose your index k to start at $k=1$, then the first term will have $(-1)^1$, and in general, the k^{th} term will have $(-1)^k$. Furthermore, notice that every term, besides the alternating sign, is a multiple of $\frac{1}{5}$. We then write,

$$-\frac{1}{5}+\frac{2}{5}-\frac{3}{5}+\frac{4}{5}-\frac{5}{5}=\sum_{k=1}^{5}(-1)^{k}\frac{k}{5}.$$

Certainly, you can do this many other ways. One sensible way is that you factor out the $-\frac{1}{5}$ first, which means

$$-\frac{1}{5}+\frac{2}{5}-\frac{3}{5}+\frac{4}{5}-\frac{5}{5}=-\frac{1}{5}(1-2+3-4+5).$$

Then, you deal with the sum in the bracket.

$$-\frac{1}{5}(1-2+3-4+5)=-\frac{1}{5}\sum_{k=1}^{5}(-1)^{k+1}k$$

where you realise that the sum in the bracket starts with a positive sign, so the associated power of -1 is $k+1$, given that you start your sum with $k=1$. You may write this also as

$$-\frac{1}{5}\sum_{k=1}^{5}(-1)^{k+1}k=-\frac{1}{5}\sum_{k=0}^{4}(-1)^{k+2}(k+1)$$

by shifting to the left. At the same time, $(-1)^{k+2}=(-1)^k(-1)^2=(-1)^k$ so you may simplify a little more.

Problem 3. (Limits of Finite Sums) For the functions below, find a formula for the finite sum (Riemann sum) obtained by dividing the interval $[a,b]$ into n equal subintervals and use the **right-hand endpoints** of each subinterval. Then take a limit as $n\to\infty$ to calculate the area under the curve over $[a,b]$.

(1) $f(x)=2x$ over $[0,3]$.

Solution. Always do the following:

(a) Subinterval length:

$$\Delta x=\frac{3-0}{n}=\frac{3}{n}.$$

(b) Write down the partition. It contains all the points you will possibly evaluate with f.

$$P=\left\{0,\frac{3}{n},\frac{6}{n},\frac{9}{n},\ldots,\frac{3n}{n}\right\}$$

noting that the last term is 3, indeed our right endpoint.

(c) Form the total area of rectangles. It is an expression of n. Using the right endpoint, we start at $\frac{3}{n}$ and end at $\frac{3n}{n}$. For the sigma notation, you should always try starting the index with $k = 1$ and end at n, since you are using n subintervals (so n rectangles to add up, i.e. your summand should have n terms).

$$\begin{aligned}
Area\,(n) &= \Delta x f\left(\frac{3}{n}\right) + \Delta x f\left(\frac{6}{n}\right) + \cdots + \Delta x f\left(\frac{3n}{n}\right) \\
&= \Delta x \left(f\left(\frac{3}{n}\right) + f\left(\frac{6}{n}\right) + \cdots + f\left(\frac{3n}{n}\right)\right) \\
&= \boxed{\frac{3}{n}\sum_{k=1}^{n} f\left(\frac{3k}{n}\right)} \\
&= \frac{3}{n}\sum_{k=1}^{n} 2\frac{3k}{n} \\
&= \frac{18}{n^2}\sum_{k=1}^{n} k \\
&= \frac{18}{n^2}\frac{n\,(n+1)}{2} \\
&= \frac{9\,(n+1)}{n}
\end{aligned}$$

At the boxed step, you could do sanity check by plugging in $k = 1$ and see if you retrieve the first term you wanted. Make sure you write down the sum without sigma notation first, as it is hard to mess up there. Converting into sigma notation takes some practice.

(d) Now, as we take a limit $n \to \infty$, we find

$$\lim_{n\to\infty} Area\,(n) = \lim_{n\to\infty} \frac{9\,(n+1)}{n} = 9 \lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = 9.$$

(2) $f\,(x) = 3x + 2x^2$ over $[0, 1]$.

(a) Subinterval length:

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}.$$

(b) Write down the partition. It contains all the points you will possibly evaluate with f.

$$P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}\right\}$$

noting that the last term is 1, indeed our right endpoint of the interval.

(c) Form the total area of rectangles. It is an expression of n. Using the right endpoint, we start at $\frac{1}{n}$ and end at $\frac{n}{n}$. So,

$$\begin{aligned}
Area\,(n) &= \Delta x f\left(\frac{1}{n}\right) + \Delta x f\left(\frac{2}{n}\right) + \cdots + \Delta x f\left(\frac{n}{n}\right) \\
&= \Delta x \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \\
&= \frac{1}{n}\sum_{k=1}^{n}\left[3\left(\frac{k}{n}\right) + 2\left(\frac{k}{n}\right)^2\right] \\
&= \frac{1}{n}\left[\left(\frac{3}{n}\sum_{k=1}^{n} k\right) + \frac{2}{n^2}\sum_{k=1}^{n} k^2\right] \\
&= \frac{1}{n}\left[\frac{3}{n}\frac{n\,(n+1)}{2} + \frac{2}{n^2}\frac{n\,(n+1)\,(2n+1)}{6}\right] \\
&= \frac{3\,(n+1)}{2n} + \frac{(n+1)\,(2n+1)}{3n^2}
\end{aligned}$$

(d) Lastly, we take a limit as $n \to \infty$,

$$\begin{aligned}
Area\,(n) &= \lim_{n\to\infty}\left(\frac{3\,(n+1)}{2n} + \frac{(n+1)\,(2n+1)}{3n^2}\right) \\
&= \boxed{\lim_{n\to\infty}\left(\frac{3n+3}{2n} + \frac{2n^2+3n+1}{3n^2}\right)} \\
&= \frac{3}{2} + \frac{2}{3} \\
&= \frac{13}{6}.
\end{aligned}$$

where we realised at the boxed step that the numerator and denominator have the same power of n, and therefore, as $n \to \infty$, the limit is the ratio of the coefficient of the highest power.