MATH 111-007 RECITATION 1117

Problem 1. Evaluate the following limits.

Hint: Indeterminate forms also include "1^∞, 0^0 and ∞^0". You can use L'Hopital when you get indeterminate forms like these when directly plugging in. Remember the logarithmic trick.

(1) $\lim_{x\to 0^+} x^x$.

Solution. Let $f(x) = x^x$. Then we use the logarithmic trick, we just need

$$L = \lim_{x\to 0^+} \ln(f(x)) = \lim_{x\to 0^+} \frac{\ln(x)}{x} = -\infty$$

because $\ln(x) \to -\infty$ and $\frac{1}{x} \to \infty$ as $x \to 0^+$ (so $-\infty \cdot \infty = -\infty$). Hence,

$$\lim_{x\to 0^+} f(x) = e^L = 0.$$

Remark. Technically, $L = -\infty$ is not a legitimate statement because we can't really equate anything to ∞. The more proper way of writing the above is

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln(f(x))} = e^{\lim_{x\to 0^+} \frac{\ln(x)}{x}} = 0$$

since we found $\lim_{x\to 0^+} \frac{\ln(x)}{x} = -\infty$.

(2) $\lim_{h\to 0} \frac{e^h - (1+h)}{h^2}$.

Solution.

$$\lim_{h\to 0} \frac{e^h - (1+h)}{h^2} = \lim_{h\to 0} \frac{e^h - 1 - h}{h^2} \overset{\text{“}\frac{0}{0}\text{”,L'H.}}{=} \lim_{h\to 0} \frac{e^h - 1}{2h} \overset{\text{“}\frac{0}{0}\text{”,L'H.}}{=} \lim_{h\to 0} \frac{e^h}{2} = \frac{1}{2}.$$

Problem 2. Show that there is exactly one root of the equation $e^x + x^3 = 0$.

Proof. We proceed with a two-part argument. Define $f(x) = e^x + x^3$.

(1) At least one root.

First, we find that $f(0) = 1$ while $f(-1) = e^{-1} - 1 < 0$. By the intermediate value theorem, this means f crosses the x-axis somewhere on $[-1, 0]$, that is, there exists at least one $c \in [-1, 0]$ such that $f(c) = 0$.

(2) At most one root.

Suppose there is another root $x = d$ in addition to $x = c$, say $d > c$. Then, by Rolle's theorem, there exists a point $y \in (c, d)$ such that

$$f'(y) = 0.$$

However, $f'(x) = e^x + 3x^2 > 0$ for all x. Thus $f'(y) = 0$ is unattainable, which deems our assumption of having the extra root at $x = d$ false.

Combining 1 and 2, we obtain the original claim. □

Problem 3. Sketch the function $f(x) = x(x-2)^2$.

Solution. Domain is the whole real line $(-\infty, \infty)$. x-intercepts are $x = 0$ and $x = 2$. y-intercept is $(0, 0)$. No asymptotes of any kind.

First derivative

$$f'(x) = (x-2)^2 + 2x(x-2) = (x-2)(x-2+2x) = (x-2)(3x-2)$$

which implies the critical points are at

$$x = 2, \quad x = \frac{2}{3}.$$

We find that

	$\left(-\infty, \frac{2}{3}\right)$	$\left(\frac{2}{3}, 2\right)$	$(2, \infty)$
$f'(x)$	$+$	$-$	$+$
$f(x)$	inc.	dec.	inc.

Next, concavity,

$$f''(x) = 3x - 2 + 3(x-2) = 3x - 2 + 3x - 6 = 6x - 8.$$

So, we only have one possible inflection point at $x = \frac{4}{3}$.

	$\left(-\infty, \frac{4}{3}\right)$	$\left(\frac{4}{3}, \infty\right)$
$f''(x)$	$-$	$+$
$f(x)$	concave down	concave up

Problem 4. State the **hypothesis** and **conclusion** of the Mean Value Theorem (MVT). Verify that the function $f(x) = \frac{1}{x-1}$ satisfies the **hypothesis** of MVT on the interval $[2, 5]$, and find all values of c in this interval that satisfy the conclusion of the theorem.

Solution. We find that $f(x)$ is continuous function on $[2, 5]$ (vertical asymptote at $x = 1$ but it's outside the domain). It is also differentiable since it is a quotient of differentiable functions. The conclusion of MVT states that the average rate of change is achieved at least somewhere on the interval. We first find the average rate of change,

$$\frac{f(5) - f(2)}{5 - 2} = \frac{\frac{1}{5-1} - \frac{1}{2-1}}{5-2} = \frac{\frac{1}{4} - 1}{3} = -\frac{1}{4}.$$

MVT states that there exists $c \in [2, 5]$ such that

$$f'(c) = -\frac{1}{4}$$

and we find

$$f'(x) = -\frac{1}{(x-1)^2} \implies -\frac{1}{(c-1)^2} = -\frac{1}{4} \implies c - 1 = \pm 2 \implies c = 3, -1.$$

We reject $c = -1$ since it is not in $[2, 5]$. Therefore, at $x = 3$, we achieve $f'(3) = -\frac{1}{4}$, that is, the average rate of change $\left(-\frac{1}{4}\right)$ is achieved at $x = 3$.

Problem 5. Find the open intervals on which the function $f(x) = x - 6\sqrt{x-1}$ is increasing and on which it is decreasing.

Solution. Note that the domain of the function is $x \geq 1$ because the square root prohibits negative inputs. Compute the first derivative and find that the critical points satisfy

$$0 = f'(x) = 1 - \frac{6}{2\sqrt{x-1}} = 1 - \frac{3}{\sqrt{x-1}}$$

We solve this equation rearranging

$$1 = \frac{3}{\sqrt{x-1}} \implies \sqrt{x-1} = 3 \implies x - 1 = 9 \implies x = 10$$

and also $x = 1$ since it makes the derivative undefined. Thus, our intervals of interest are $(1, 10)$ and $(10, \infty)$.

On $(1, 10)$, $\sqrt{x-1} < 3$ and thus $\frac{3}{\sqrt{x-1}} > 1$, and hence $1 - \frac{3}{\sqrt{x-1}} < 0$. Therefore, $f(x)$ is decreasing on $(1, 10)$.

On $(10, \infty)$, $\sqrt{x-1} > 3$ and thus $\frac{3}{\sqrt{x-1}} < 1$, and hence $1 - \frac{3}{\sqrt{x-1}} > 0$. Therefore, $f(x)$ is increasing on $(10, \infty)$.

Problem 6. Apply Newton's method to the function in Problem 2. Supply your own guess (based on where the root may be). Compute the second iteration x_2 (leave your answer in the most simplified form, however ugly it may seem).

Solution. We find

$$f'(x) = e^x + 3x^2.$$

We have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{e^{x_n} + x_n^3}{e^{x_n} + 3x_n^2}.$$

Thus, since we located the root lies on $[-1,0]$, we can start by guessing with $x_0 = 0$.

$$x_1 = x_0 - \frac{e^{x_0} + x_0^3}{e^{x_0} + 3x_0^2} = 0 - \frac{e^0 + 0^3}{e^0 + 3 \cdot 0^2} = -1$$

and then

$$x_2 = x_1 - \frac{e^{x_1} + x_1^3}{e^{x_1} + 3x_1^2} = \boxed{-1 - \frac{e^{-1} - 1}{e^{-1} + 3}}.$$

One should be able to show that $x_2 \in [-1,0]$ which means you are searching in the right space.

Problem 7. (Challenge problem, do only if you have time – this level of difficulty is unlikely but not unexpected)

A certain apartment complex has three hundred (essentially identical) apartments. The landlord knows that at a rent of one thousand dollars per month every unit will be let, but for each ten dollar rise in the rent, one more unit will go vacant. How much should the landlord charge to maximize his income?

Solution. We know the income I is rent R times number of unit let N, i.e.

$$I = RN.$$

Let x be the amount excess of \$1000. Rent is allowed to increase in the linear fashion

$$R(x) = 1000 + x$$

while for the same x, number of unit let decreases

$$N(x) = 300 - \frac{x}{10}.$$

(Technically, $\frac{x}{10}$ should be written as $\lfloor \frac{x}{10} \rfloor$, the floor function, that is, it rounds down to an integer. For example, if you increase by 9 dollars, you won't lose a unit) Altogether, we have

$$I(x) = (1000 + x)\left(300 - \frac{x}{10}\right),$$

which we now maximize on the domain $x \in [0, 3000]$ (why? You would hate to see negative income, right?).

The critical points satisfy

$$0 = I'(x) = \left(300 - \frac{x}{10}\right) + (1000 + x) \cdot \left(-\frac{1}{10}\right) = 300 - \frac{x}{10} - 100 - \frac{x}{10}$$

which implies

$$x = 1000.$$

Thus, we evaluate at the endpoints and the critical point and compare

$$I(0) = 300000$$
$$I(3000) = 0$$
$$I(1000) = 400000$$

which makes $I(1000)$ a local maximum (and thus a global one since there is only one critical point). Therefore, the landlord should charge \$2000 per unit.