MATH 111-007 RECITATION 1027

The following problems don't necessarily have to do with Chapter 4. **Any answer without an attempted explanation (when requested) receives automatically -1 for the recitation.**

Problem 1. (Odd and even functions and their derivatives)

(1) An odd function $f(x)$ satisfies $f(x) = -f(-x)$. Suppose further that f is differentiable. What can you say about $f'(x)$?

Remark. You take a derivative of both sides of the equation satisfied by any odd function.

$$\begin{aligned} f(x) &= -f(-x) \\ \implies \frac{d}{dx} f(x) &= \frac{d}{dx}(-f(-x)) \\ \overset{\text{Chain rule}}{\implies} f'(x) &= -f'(-x)\frac{d}{dx}(-x) = f'(-x). \end{aligned}$$

This means $f'(x)$ is an even function, or more precisely, the derivative of an odd function is an even function.

(2) An even function $f(x)$ satisfies $f(x) = f(-x)$. Suppose further that f is differentiable. What can you say about $f'(x)$?

Remark. Similar to the above procedure, we do

$$\begin{aligned} f(x) &= f(-x) \\ \implies \frac{d}{dx} f(x) &= \frac{d}{dx}(f(-x)) \\ \overset{\text{Chain rule}}{\implies} f'(x) &= f'(-x)\frac{d}{dx}(-x) = -f'(-x). \end{aligned}$$

This means $f'(x)$ is an odd function, or more precisely, the derivative of an even function is an odd function.

Problem 2. Show that $y_1 = \sin(x)$, $y_2 = \cos(x)$ and $y_3 = a\cos(x) + b\sin(x)$ where a and b are constants, all satisfy the equation

$$y'' + y = 0.$$

How would you modify the functions above so that they satisfy $y'' + 4y = 0$? How about $y'' + ky = 0$ for $k \neq 0$?

Solution. You plug the given functions into the equation and see if they work.

For $y_1 = \sin(x)$, we find $y_1'' = -\sin(x)$ and thus $y_1'' + y_1 = 0$, check.
For $y_2 = \cos(x)$, we find $y_2'' = -\cos(x)$ and thus $y_2'' + y_2 = 0$, check.
Lastly, for $y_3 = a\cos(x) + b\sin(x) = ay_2 + by_1$, we find

$$y_3'' = ay_2'' + by_1''$$

and therefore,

$$\begin{aligned} y_3'' + y_3 &= ay_2'' + by_1'' + ay_2 + by_1 \\ &= a(y_2'' + y_2) + b(y_1'' + y_1) \\ &= 0, \end{aligned}$$

check!

Now, what would contribute to the 4 out there? Suppose $y_1 = \sin(2x)$ now,

$$y_1' = 2\cos(2x), \quad y_1'' = -4\sin(2x)$$

and therefore

$$y_1'' + 4y_1 = 0,$$

which means that we can deal with the 4 by scaling the input (thanks to the Chain Rule). Therefore,

$$y_2 = \cos(2x), \quad y_3 = a\cos(2x) + b\sin(2x)$$

will also work. In general, to satisfy $y'' + ky = 0$, we just need the scale of input to be $\sqrt{k}$, that is,

$$y_3 = a\cos\left(\sqrt{k}x\right) + b\sin\left(\sqrt{k}x\right).$$

Problem 3. Consider the function $f(x) = x^3$. Construct an interval such that

(1) both absolute extrema exist. Give their values.
   Any closed and bounded intervals will work because f is a continuous function (this is the consequence of the Extreme Value Theorem). For example, $[0,1]$ works, and absolute max is at $(1,1)$ while absolute min is at $(0,0)$.
(2) only absolute minimum exists, but not absolute maximum.
   Any half open interval with the open end on the right will work (because the function is monotone increasing), e.g. $[0,1)$. Absolute min is at $(0,0)$ but no absolute max exists.
(3) no absolute extrema.
   $(0,1)$.

Problem 4. Again, consider the function $f(x) = x^3$. Is there a local maximum at $x = 0$? Why or why not?
Support your claims by using the definition of local maximum directly, e.g. if you claim it is a local maximum, then you need to tick all the check points of the definition; on the other hand, if you claim it is not, then state exactly which check points have failed in the definition in this situation.

Solution. It is NOT a local maximum. Any open intervals (a,b) containing $x = 0$ has to go into both the positive and negative, that is, $a < 0$ and $b > 0$. Clearly, $f(x) \geq f(0)$ for $0 \leq x < b$. So there is no open interval (a,b) (neighbourhood) containing $x = 0$ such that $f(0) \geq f(x)$ for all $x \in (a,b)$.

Theorem. *(Extreme Value Theorem) If f is continuous on a closed and bounded interval $[a,b]$, then f achieves both its absolute maximum and minimum on $[a,b]$. More precisely, there are numbers $x_1, x_2 \in [a,b]$ such that $f(x_1) = m$ and $f(x_2) = M$, and*

$$m \leq f(x) \leq M, \quad \textit{for all } x \in [a,b].$$

One way to read a theorem is via the implication "If A then B". The contrapositive statement "If not B then not A" is an equivalent statement. To check consistency, we can check either statement.

Regarding the negation "not A" or "not B", if A contains two statements, then "not A" means either statement is untrue (or both). Therefore, the contrapositive of the extreme value theorem is the following:

Theorem. *(Contrapositive of Extreme Value Theorem) Suppose $D = [a,b]$ is closed an bounded. If f does not have **both** absolute extrema on $[a,b]$ (so having at most one extremum), then f is discontinuous on $[a,b]$.*

Problem 5. (Sketch and explain) For the following functions, identify the domain D. Determine whether $f(x)$ achieves absolute extrema on D. Explain how your answer is consistent with the Extreme Value Theorem (given in lecture and above).

(1) $f(x) = \begin{cases} x+1, & -1 \leq x < 0; \\ \cos(x), & 0 < x \leq \frac{\pi}{2}. \end{cases}$

   Solution. First, in order to apply the Extreme Value Theorem, one must show that
   (a) $f(x)$ is continuous.
   (b) The domain is closed and bounded.

   The domain $D = \left[-1, \frac{\pi}{2}\right]$ is indeed closed and bounded.
   Now, onto continuity, the point of controversy is $x = 0$, as both pieces are continuous in their own domains. Note, however, that $x = 0$ is never included in the domain, which means, the function

cannot be continuous on D. Therefore, the direct statement of the theorem does not even apply. We go on to check the contrapositive.

We find the absolute minimum is at $(-1,0)$ and $\left(\frac{\pi}{2},0\right)$ and no absolute maximum. This satisfies the hypothesis of the contrapositive statement. Therefore, f must be discontinuous on $[a,b]$, which is the case as we found above. Therefore, the results are consistent with the theorem.

(2) $f(x)=\begin{cases}\frac{1}{x}, & -1\le x<0;\\ \sqrt{x}, & 0\le x\le 4.\end{cases}$

Solution. The apparent asymptote implies that $f(x)$ is not continuous, though the domain we work with, $[-1,4]$, is closed and bounded. We check consistency using the contrapositive statement.

Indeed, we find only absolute maximum at $x=4$ (the coordinate $(4,2)$), and no absolute minimum. This satisfies the hypothesis of the contrapositive statement, which then implies f must be discontinuous. This is consistent with our finding.

(3) $f(x)=\begin{cases}1, & x=0;\\ 0, & -1\le x<0 \text{ and } 0<x\le 1.\end{cases}$

Solution. The function has a jump at $x=0$, and thus is discontinuous. The domain is closed and bounded $[-1,1]$. The forward statement is not applicable, and we can conclude nothing. We look at the contrapositive statement.

We actually find both absolute maximum and minimum, at $(0,1)$ and $(-1,0)$ (actually all of $-1\le x<0$ and $0<x\le 1$ achieves absolute minima). This means that the hypothesis of the contrapositive statement is not satisfied, and we can conclude nothing.

So what went wrong? Is the theorem wrong? No, if you find an inapplicable example (in any way), it does NOT invalidate the theorem. It showcases the limitation of the theorem, usually given by its hypothesis that the theorem does not apply to ALL but some functions. If you really want to disprove a theorem, you must pick an example that satisfies the hypothesis, and then come up with a different conclusion than the promised one.