MATH 111-107 RECITATION 1013

Warm-up

Problem 1. Evaluate the following items:

(1) $5^{\log_5 100} = 100$.
(2) $5^{2\log_5 100} = 5^{\log_5 100^2} = 100^2$.
(3) $\log_5 25 = 2$.
(4) $\log_4 \frac{1}{16} = \log_4 (4)^{-2} = -2$.
(5) $\log_4 2 + \log_4 32 = \log_4 (2 \cdot 32) = \log_4 64 = 3$.
(6) $\frac{\log_3 25}{\log_3 5} = \log_5 25 = 2$.

Problem 2. Find the domain and then derivative of the following functions on their respective domain:

(1) $\ln(\sec(x))$.

Solution. $\ln(x)$ takes only $x > 0$. So here we must require that $\sec(x) > 0$. Equivalent, $\frac{1}{\cos(x)} > 0 \implies \cos(x) > 0$. Which angles give you positive cosine values? $-\frac{\pi}{2} < x < \frac{\pi}{2}$ (and π-periodic repetitions of interval). By Chain rule, we note that the outer function is $\ln(x)$ and the inner function is $\sec(x)$.

$$\frac{d}{dx}(\ln(\sec(x))) = \frac{1}{\sec(x)}\frac{d}{dx}(\sec(x)) = \frac{1}{\sec(x)}\sec(x)\tan(x) = \tan(x).$$

Remark. Now, when someone asks you, which function's derivative gives you $\tan(x)$, you can slap $\ln(\sec(x))$ in their face.

(2) $(\ln(5x))^3$.

Solution. Domain is $x > 0$.

$$\frac{d}{dx}(\ln(5x))^3 = 3(\ln(5x))^2\frac{d}{dx}\ln(5x) = 3(\ln(5x))^2\frac{1}{x} = \frac{3(\ln(5x))^2}{x}$$

where we used the result $\frac{d}{dx}\ln(bx) = \frac{1}{x}$, independent of b.

Problem 3. A big block of ice is in the shape of a perfect cube. As it melts, each edge of the cube is decreasing at the rate of 2 cm/min. Suppose you observe that the edge of the ice cube is 80 cm, at what rate is the ice cube's

(1) surface area changing?

Solution. Let $A(s)$ be the surface area of a cube with side length s. Then,

$$A(s) = 6s^2.$$

Now, we know the information $\frac{ds}{dt} = -2$ cm/min (decreasing).

$$\frac{dA}{dt} = \frac{dA}{ds}\frac{ds}{dt} = 12s(t)\cdot(-2) = -24s(t).$$

When $s(t) = 80$, $\frac{dA}{dt}|_{s=80} = -24 \cdot 80 = -1920\ inch^2/\text{min}$.

(2) volume changing?

Solution. Let $V(s)$ be the volume of a cube with side length s. Then,

$$V(s) = s^3.$$

We then find

$$\frac{dV}{dt} = \frac{dV}{ds}\frac{ds}{dt} = 3s^2(-2) = -6s^2.$$

Thus,

$$\frac{dV}{dt}\mid_{s=80}= -6\,(80)^2 = -25600\,inch^3/min.$$

Problem 4. Consider an isosceles triangle with base length 10 inches and side length x inches. The side length is increasing at a rate of 4 inches/min. When the side length is 13 inches, at what rate is the triangle's

(1) perimeter changing?

Solution. Let $P\,(t)$ be the perimeter of the isosceles at time t. Then

$$P = 2x + 10.$$

We know $\frac{dx}{dt} = 4$ inches/min. We want $\frac{dP}{dt}\mid_{x=13}$.

$$\frac{dP}{dt} = \frac{dP}{dx}\frac{dx}{dt} = 2\cdot 4 = 8.$$

Thus $\frac{dP}{dt}\mid_{x=13}= 8$ inches/min, that is, independent of what the side length is.

(2) height changing?

Solution. Let $h\,(t)$ be the perimeter of the isosceles at time t. Then

$$h^2 + 5^2 = x^2.$$

We want $\frac{dh}{dt}\mid_{x=13}$ and know again $\frac{dx}{dt} = 4$ inches/min. Thus, taking a derivative $\frac{d}{dt}$ on both sides, we have

$$2h\,(t)\,\frac{dh}{dt} = 2x\frac{dx}{dt} \implies \frac{dh}{dt} = \frac{x\,(t)}{h\,(t)}\frac{dx}{dt} = \frac{4x\,(t)}{h\,(t)}$$

Now, when $x = 13$, $h = \sqrt{13^2 - 5^2} = 12$. Thus,

$$\frac{dh}{dt}\mid_{x=13}= \frac{4\cdot 13}{12} = \frac{13}{3}\,inch/min.$$

(3) area changing?

Solution. Let $A\,(t)$ be the area of the isosceles at time t. Then

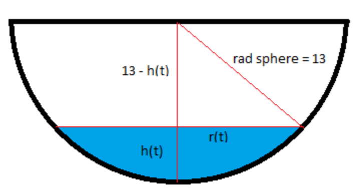
$$A = \frac{1}{2}10h = 5h.$$

We want $\frac{dA}{dt}\mid_{x=13}$.

$$\frac{dA}{dt} = 5\frac{dh}{dt} = 5\frac{13}{3} = \frac{65}{3}.$$

Problem 5. An open hemispherical tank has radius 13 feet. Oil begins flowing into the tank in such a way that the depth h of the oil in the tank changes at the rate of 3 ft/hr. At what rate is the top circular surface area of the oil changing when the depth of oil is

(1) $h = 1$ ft?
(2) $h = 8$ ft?
(3) Which one is bigger? Does it surprise you or does it make sense?



Solution. Let S be the top circular surface area. How can S be related to oil depth h? Consider $r\,(t)$ the radius of the surface circle at time t, we clearly see that

$$r^2 + (13 - h\,(t))^2 = 13^2.$$

Now, what is the area then?

$$S = \pi r^2 = \pi\left(13^2 - (13-h)^2\right) = \pi\left(26h - h^2\right)$$

which now is conveniently a function depth h only. We also know $\frac{dh}{dt} = 3$ ft/hr. Thus,

$$\begin{aligned}\frac{dS}{dt} &= \frac{dS}{dh}\frac{dh}{dt} \\ &= (26\pi - 2\pi h)\cdot 3\end{aligned}$$

We want first

$$\frac{dS}{dt}\,|_{h=1} = (26\pi - 2\pi)\cdot 3 = 42\pi$$

and second

$$\frac{dS}{dt}\,|_{h=8} = (26\pi - 2\pi\cdot 8)\cdot 3 = 30\pi$$

So, when the oil is only 1 foot deep, the rate of area change is bigger than when the oil is very deep at 8 feet. It should make sense because if we consider the same amount of oil dumped into the tank, it is relatively harder to gain depth when it is deep already – and depth directly influences the surface area.

Problem 6. (A sliding ladder) A ladder of constant length L is leaning against the wall. Its base is sliding away from the wall at some rate c ft/min. Find the rate at which the tip of the ladder is sliding down the wall when it hits the ground.

Solution. Let $x(t)$ be the distance of the base of the ladder to the wall, and $y(t)$ be the height of the ladder on the wall. Then

$$x^2 + y^2 = L^2.$$

Taking a derivative with respect to time, we have

$$2xx'(t) + 2yy'(t) = 0 \implies y'(t) = -\frac{xx'(t)}{y(t)} = -\frac{cx(t)}{\sqrt{L^2 - x^2(t)}}.$$

When the ladder hits the ground, we must have $x(t) = L$ since all of the ladder is the on the ground. But you quickly realise that setting $x(t) = L$ gives you 0 in the denominator for $y'(t)$. Therefore, you must do a limit process

$$\lim_{x\to L} y'(t) = -\lim_{x\to L}\frac{cx(t)}{\sqrt{L^2 - x^2(t)}} = -\infty.$$

This means the moment the ladder hits the ground, we got **infinite** velocity.

Problem. (Bonus) Is the previous problem physically sound? If so, does the result make sense? If not, what assumption did we implicitly make to make the problem "work"? What really should happen when something is sliding down the wall?

Go read this paper: https://www.maa.org/sites/default/files/0746834218966.di020770.02p0200y.pdf.

It shouldn't make sense because when the ladder falls, the tip also gets pushed off the wall – so automatically we lose the meaning of $x(t)$. Why does the ladder get pushed off the wall? You go figure it out in your physics 1 class when you learn about torque. In essence, there is a critical angle (as a function of the pulling speed) that the ladder will lose contact off the wall.