MATH 111-007 RECITATION 5

OCTOBER 6TH, 2021

(1) The Product Rule operates on a product and yields a sum.

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

(2) The Chain Rule operates on a composite function and yields a product.

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x).$$

Problem 1. Find the derivative of the following functions. If you scored lower than a 6 in Quiz 4, please write out the inner and outer functions, and do it properly using the two rules outlined above.

(1) $h(z) = \sin\left(\frac{1}{z-1}\right)$.

Solution. Inner function is $g(z) = \frac{1}{z-1}$, while the outer is $f(z) = \sin(z)$. Thus,

$$h'(z) = \frac{d}{dz}h(z) = \frac{d}{dz}f(g(z)) = f'(g(z))g'(z)$$

We find

$$f'(z) = \cos(z) \implies f'(g(z)) = \cos\left(\frac{1}{z-1}\right),$$

and

$$g'(z) = \frac{d}{dz}(z-1)^{-1} = -(z-1)^{-2}.$$

Altogether,

$$h'(z) = -\frac{\cos\left(\frac{1}{z-1}\right)}{(z-1)^2}.$$

(2) $p(q) = e^{\frac{1}{q^2}}$.

Solution. You can do this quickly by the exponential chain rule. But if you want to find out what's really going on without relying on formulas too much, we find that the inner function is $g(q) = \frac{1}{q^2}$, while the outer is $f(q) = e^q$. Thus,

$$p'(q) = \frac{d}{dq}p(q) = \frac{d}{dq}f(g(q)) = f'(g(q))g'(q)$$

We find

$$f'(q) = e^q \implies f'(g(z)) = e^{\frac{1}{q^2}},$$

and

$$g'(q) = \frac{d}{dq}q^{-2} = -2q^{-3}.$$

Altogether,

$$p'(q) = -\frac{2e^{\frac{1}{q^2}}}{q^3}.$$

(3) $g(t) = \cos^2\left(4\sqrt{t}\right)$.

Solution. Similar to the quiz question, this one has three layers. First, inner $h(t) = 4\sqrt{t}$ and outer $f(t) = \cos^2(t)$ .Thus,

$$g'(t) = \frac{d}{dt} f(h(t)) = f'(h(t))\, h'(t).$$

Now,

$$f'(t) = 2\cos(t)(-\sin t)$$

invokes the Chain Rule also (but very simply). Thus,

$$f'(h(t)) = -2\sin\left(4\sqrt{t}\right)\cos\left(4\sqrt{t}\right) = -\sin\left(8\sqrt{t}\right)$$

where the last equality is optional (using the double angle formula $\sin(2x) = 2\sin(x)\cos(x)$). Also,

$$h'(t) = 4\frac{d}{dt}t^{\frac{1}{2}} = 4 \times \frac{1}{2}t^{-\frac{1}{2}} = \frac{2}{\sqrt{t}}.$$

Altogether,

$$g'(t) = \frac{-4\sin\left(4\sqrt{t}\right)\cos\left(4\sqrt{t}\right)}{\sqrt{t}} = -\frac{2\sin\left(8\sqrt{t}\right)}{\sqrt{t}}$$

both are acceptable answers.

(4) $f(x) = \left(\frac{1+\sin(3x)}{3-2x}\right)^{-1}$.

Solution. First thing to do is not the power chain rule on the outer most shell. You just work out the -1 power.

$$f(x) = \frac{3-2x}{1+\sin(3x)}.$$

Then, simply, quotient rule,

$$\begin{aligned} f'(x) &= \frac{(-2)(1+\sin(3x)) - (3-2x)\cos(3x)\cdot 3}{(1+\sin(3x))^2} \\ &= -\frac{2}{1+\sin(3x)} - \frac{3(3-2x)\cos(3x)}{(1+\sin(3x))^2} \end{aligned}$$

Problem 2. Find the equation of the tangent line to $y = \sqrt{x^2 - x + 7}$ at $x = 2$. In addition, find the equation of the normal (perpendicular to the tangent) at the same point.

Solution. Slope of the tangent line at $x = 2$ is $y'(2)$. Write $y = \left(x^2 - x + 7\right)^{\frac{1}{2}}$. Then by the Power Chain Rule, we have

$$y'(x) = \frac{1}{2}\left(x^2 - x + 7\right)^{-\frac{1}{2}}(2x - 1)$$

and therefore,

$$y'(2) = \frac{1}{2}\left(2^2 - 2 + 7\right)^{-\frac{1}{2}}(2\cdot 2 - 1) = \frac{1}{2}\cdot 9^{-\frac{1}{2}}\cdot 3 = \frac{1}{2}.$$

To get the equation of the tangent line, we need a point. We do, for $x = 2$, $y(2) = 3$. Therefore, the equation of the tangent line is

$$y_{\text{tangent line}} - 3 = \frac{1}{2}(x - 2).$$

The normal is perpendicular to the tangent line, and thus it has slope the negative reciprocal of that of the tangent, namely, -2. It passes through the point of tangency to the curve $(2, 3)$, and thus

$$y_{\text{normal}} - 3 = -2(x - 2).$$

Problem 3. Find $\frac{d^{111}y}{dx^{111}}$ for $y = xe^x$.

Solution. When you see a problem like this (this is one of the past exam problems), you just start with one derivative and then two and then try to see a pattern. By the product rule,

$$\frac{dy}{dx} = e^x \frac{d}{dx}(x) + x\frac{d}{dx}(e^x) = e^x + xe^x = (1+x)\,e^x.$$

Then, by the product rule again, we see similarly that

$$\frac{d^2y}{dx^2} = e^x \frac{d}{dx}(1+x) + (1+x)\frac{d}{dx}(e^x) = e^x + (1+x)\,e^x = (2+x)\,e^x.$$

See a pattern now? It seems you always get $(n+x)\,e^x$ where n is the number of derivatives you take. Thus, you can now say that

$$\frac{d^{111}y}{dx^{111}} = (111+x)\,e^x.$$

(A real proof involves the mathematical induction, which is not required by the class material. The above rationale would form the correct solution to write.)

Problem 4. Find $\frac{dy}{dx}$ for $y^2 = x$.

Solution. Differentiate both sides. By Chain rule,

$$\begin{aligned} \frac{d}{dx}\left(y^2\right) &= \frac{d}{dx}(x) \\ \implies 2y\frac{dy}{dx} &= 1 \\ \implies \frac{dy}{dx} &= \frac{1}{2y}. \end{aligned}$$

Done.

Problem 5. Find the equation of the tangent line of the curve $x^2y^2 = 9$ at $(-1, 3)$.

Solution. We need $\frac{dy}{dx}$ and then evaluate it at $x = -1, y = 3$. Differentiating both sides,

$$\begin{aligned} \frac{d}{dx}\left(x^2y^2\right) &= \frac{d}{dx}(9) \\ \implies 2xy^2 + x^2\left(2y\frac{dy}{dx}\right) &= 0 \\ \implies 2x^2y\frac{dy}{dx} &= -2xy^2 \\ \implies \frac{dy}{dx} &= -\frac{xy^2}{x^2y} \end{aligned}$$

If you want to cancel, you must say, when $x \neq 0$ and $y \neq 0$, then,

$$\frac{dy}{dx} = -\frac{y}{x}$$

which also only holds for $x \neq 0$. Altogether, the slope at $(-1, 3)$ is

$$\frac{dy}{dx}\,|_{x=-1,y=3} = -\frac{3}{(-1)} = 3$$

Then, using the point at $(-1, 3)$, we have the tangent line by point-slope form,

$$y_{\text{tangent line}} - 3 = 3\,(x-1)\,.$$

Problem 6. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the curve $3 + \sin y = y - x^3$. Write the solutions in terms of x and y only.

Solution. We proceed by taking a derivative of both sides of the equation.

$$\begin{aligned} \frac{d}{dx}(3+\sin y) &= \frac{d}{dx}(y-x^3) \\ \implies \cos(y)\frac{dy}{dx} &= \frac{dy}{dx} - 3x^2 \\ \implies \cos(y)\frac{dy}{dx} - \frac{dy}{dx} &= -3x^2 \\ \implies (\cos(y)-1)\frac{dy}{dx} &= -3x^2 \\ \implies \frac{dy}{dx} &= \frac{3x^2}{1-\cos(y)} \end{aligned}$$

Then, we go on to take one more derivative,

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ &= \frac{d}{dx}\left(\frac{3x^2}{1-\cos(y)}\right) \\ &= \frac{6x(1-\cos(y)) - (3x^2)\boxed{\left(\sin(y)\frac{dy}{dx}\right)}}{(1-\cos(y))^2} \end{aligned}$$

Then, you plug in $\frac{dy}{dx} = \frac{3x^2}{1-\cos(y)}$ and clean up the expression. Note that the boxed step follow from chain rule on

$$\frac{d}{dx}(1-\cos(y)) = -\frac{d}{dx}(\cos(y)) = -(-\sin(y))\frac{dy}{dx} = \sin(y)\frac{dy}{dx}.$$