MATH 111-007 RECITATION 4

A well-known result for the exponential function is

$$\frac{d}{dx}\left(e^{x}\right)=e^{x},$$

namely, the function and its derivative (and thus any order of derivatives) are equal.

Problem 1. Compute $\frac{d}{dx}\left(e^{2x}\right)$ without using the Chain Rule (which we haven't learned).

Solution.

$$\frac{d}{dx}\left(e^{2x}\right)=\frac{d}{dx}\left(e^{x}\cdot e^{x}\right)=e^{x}\left(\frac{d}{dx}e^{x}\right)+\left(\frac{d}{dx}e^{x}\right)e^{x}=e^{x}\cdot e^{x}+e^{x}\cdot e^{x}=e^{2x}+e^{2x}=2e^{2x}.$$

Problem 2. Find the derivative of $y=\frac{(x-1)\left(x^{2}-2x\right)}{x^{4}}$. Use the **best** method in your opinion (which means you should try several).

Solution. The best way is to multiply out and turn it into a sum of functions, rather than using the quotient rule.

$$y=\frac{(x-1)\left(x^{2}-2x\right)}{x^{4}}=\frac{x^{3}-x^{2}-2x^{2}+2x}{x^{4}}=\frac{x^{3}-3x^{2}+2x}{x^{4}}=\frac{1}{x}-\frac{3}{x^{2}}+\frac{2}{x^{3}}.$$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{1}{x}-\frac{3}{x^{2}}+\frac{2}{x^{3}}\right)=-\frac{1}{x^{2}}+\frac{6}{x^{3}}-\frac{6}{x^{4}}.$$

Quotient rule is of course okay, but just a lot messier.

Moral of the story: when you see rational functions like this with only one term in the denominator, it is always advisable to turn the expression into sums by the above procedure. Rational functions with more complicated denominators such as $\left(x^{3}+1\right)$ or $\left(x^{2}+x\right)$ should involve the quotient rule instead, as the above division leads nowhere.

Problem 3. Consider the free fall

$$s(t)=\frac{1}{2}gt^{2}$$

where $s(t)$ is the vertical displacement in meters, of a free falling object. g is the gravitational acceleration with unit m/s^{2}, though kept general here (so, not specifically the Earth's). Find t_{0}, the time it takes for the downward velocity of the object to reach some level v_{0}. How does t_{0} depend on g?

Solution. We know

$$v(t)=\frac{ds}{dt}=gt.$$

Then, to reach a velocity of v_{0}, we solve for t_{0} with

$$v_{0}=gt_{0}\implies t_{0}=\frac{v_{0}}{g}.$$

t_{0} in fact depends inversely on g. Now, with this in hand, you can compute any velocity on any planet with a known g.

Problem 4. Consider a projectile on an airless planet (so no air friction considered) with unknown gravitational acceleration constant g_{s}. The height of the projectile (in meters) as a function of time t follows

$$s(t)=15t-\frac{1}{2}g_{s}t^{2}.$$

The projectile reached its maximum height 20 seconds after being launched. What was the value of g_{s}?

Solution. When the projectile reached its maximum height at the 20^{th} second, its vertical velocity is 0, namely, $v(20) = 0$. We do know that

$$v(t) = \frac{ds}{dt} = 15 - g_s t.$$

With this,

$$0 = v(20) = 15 - 20g_s \implies g_s = \frac{3}{4}$$

with unit in $meter/sec^2$.

Problem 5. Determine all vertical asymptotes, and find the derivative of the following functions:

(1) $\tan(x)$. VA: $x = \frac{\pi}{2} + n\pi = \frac{(2n+1)\pi}{2}$, n an integer. $\frac{d}{dx}(\tan x) = \sec^2(x)$.
(2) $\cot(x)$. VA: $x = \pi n$, n an integer. $\frac{d}{dx}(\cot(x)) = -\csc^2(x)$.
(3) $\sec(x)$. VA: $x = \frac{\pi}{2} + n\pi$, n an integer. $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$.
(4) $\csc(x)$. VA: $x = \pi n$, n an integer. $\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$.

For $\tan(x)$ and $\cot(x)$, is there anything that can be corroborated by their derivatives and their asymptotes?

Remark. Note $\tan(x)$ and $\sec(x)$ (and thus $\sec^2(x)$) have the same set of asymptotes. So does the pair $\cot(x)$ and $\csc(x)$ (and thus $-\csc^2(x)$).

Problem 6. Find the derivative of $y = \frac{\cos^2(x)}{1+\sin(x)}$, for $x \in [-\pi, \pi]$. Determine the **best** method.

Solution. The best way is to rewrite the numerator using a trig identity.

$$y = \frac{\cos^2(x)}{1+\sin(x)} = \frac{1-\sin^2(x)}{1+\sin(x)} = \frac{(1-\sin(x))(1+\sin(x))}{1+\sin(x)}.$$

Now, can you cancel right away? You can, unless $x = -\frac{\pi}{2}$ since it makes the bottom 0. So we need to separate into two cases.

Case 1. $x \neq -\frac{\pi}{2}$.
This is straightforward now since we are away from the potentially bad point.

$$y = 1 - \sin(x) \implies y'(x) = \frac{d}{dx}(1 - \sin(x)) = -\cos(x).$$

Case 2. $x = -\frac{\pi}{2}$.
You need to check the derivative from both sides. But this is no problem since you know the behaviour of the derivative away from this point, and the left and right limits need exactly that! In fact, it is the continuity of the derivative on both sides that now gives you differentiability.

$$\lim_{x\to-\frac{\pi}{2}^-} y'(x) = \lim_{x\to-\frac{\pi}{2}^-} -\cos(x) = 0$$

and

$$\lim_{x\to-\frac{\pi}{2}^+} y'(x) = \lim_{x\to-\frac{\pi}{2}^+} -\cos(x) = 0$$

which shows the left and right derivatives are equal at $x = -\frac{\pi}{2}$.

Altogether, $\frac{dy}{dx} = -\cos(x)$.

Problem 7. Consider the piece-wise function

$$f(x) = \begin{cases} x + b, & x < 0, \\ \cos(x), & x \geq 0. \end{cases}$$

Is there a value of b that will make $f(x)$ continuous at $x = 0$? Differentiable at $x = 0$ (Think about this. Think!)? Justify.

Solution. To enforce continuity, we check left, right limit and the function value (achieved by the right limit since $x \geq 0$ is inclusive). We force

$$\lim_{x\to0^-} f(x) = \lim_{x\to0^+} f(x) = f(0)$$

which implies

$$b = \cos(0) = 1.$$

To enforce differentiability, it is necessary to have continuity, so $b = 1$ has to be true. At the same time, we also need the slopes/derivatives from left *and* right to match, that is,

$$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^+} f'(x).$$

But this would imply

$$\lim_{x \to 0^-} \frac{d}{dx}(x + b) = \lim_{x \to 0^+} \frac{d}{dx}(\cos(x))$$

and therefore

$$\lim_{x \to 0^-} 1 = \lim_{x \to 0^+} -\sin(x)$$

But, this is (obviously) trouble since $1 \neq 0$. Therefore, there is no such b that can make $f(x)$ differentiable.