MATH 111-007 QUIZ 9

Problem 1. Sketch the function $f(x) = \frac{x}{1+x^2}$.

(1) Identify the domain of f and symmetries the curve may have. Find all intercepts.

(a) Domain: $(-\infty, \infty)$. Note that the numerator is always positive.

(b) Also note that

$$f(-x) = \frac{-x}{1+(-x)^2} = -\frac{x}{1+x^2} = -f(x),$$

implying that the function is odd (symmetric about $y = x$).

(c) Intercepts: y-intercept is the value of $f(0) = 0$. x-intercept is the value x such that $f(x) = 0$. Well, you hit two birds with one stone. $(0, 0)$ is both intercepts.

(2) Identify any asymptotes that may exist.

No vertical asymptotes. There is a horizontal asymptote, justified by computing

$$\lim_{x\to\pm\infty} \frac{x}{1+x^2} = \lim_{x\to\pm\infty} \frac{\frac{1}{x}}{\frac{1}{x^2}+1} = 0.$$

Thus, $y = 0$ is the expression for the horizontal asymptote.

(3) Find the derivatives f' and f''.

You (could) do quotient rule in both derivatives. Here, I do a quotient rule for the first derivative and a product rule for the second.

$$f'(x) = \frac{\left(1+x^2\right) - x \cdot 2x}{\left(1+x^2\right)^2} = \frac{1-x^2}{\left(1+x^2\right)^2}$$

and

$$\begin{aligned}
f''(x) &= \frac{d}{dx}\left(1-x^2\right)\left(1+x^2\right)^{-2} \\
&\overset{\text{product rule}}{=} -2x\left(1+x^2\right)^{-2} + \left(1-x^2\right)(-2)\left(1+x^2\right)^{-3} 2x \\
&\overset{\text{rearrange}}{=} -\frac{2x}{\left(1+x^2\right)^2} - \frac{4x\left(1-x^2\right)}{\left(1+x^2\right)^3} \\
&\overset{\text{common denominator}}{=} -\frac{2x\left(1+x^2\right)+4x-4x^3}{\left(1+x^2\right)^3} \\
&= -\frac{2x+2x^3+4x-4x^3}{\left(1+x^2\right)^3} \\
&= -\frac{6x-2x^3}{\left(1+x^2\right)^3} \\
&\overset{\text{sign flip}}{=} \frac{2x^3-6x}{\left(1+x^2\right)^3} \\
&= \frac{2x\left(x^2-3\right)}{\left(1+x^2\right)^3}
\end{aligned}$$

(4) Find the critical points of y, if any, and identify the function's behaviour at each one.

We solve

$$0 = f'(x) = \frac{1-x^2}{\left(1+x^2\right)^2} \implies x = \pm 1.$$

Behaviour at each critical point will only come around when we do first derivative test in the next step.

(5) Find where the curve is increasing and where it is decreasing.

The critical points are used to form partitions of the domain. We have 3 separate intervals now. Note that the numerator is always positive. The sign of f' is now decided completely by the sign of $1-x^2$.

(a) $(-\infty,-1)$: $1-x^2<0 \implies f'(x)<0$, i.e. the function is decreasing.
(b) $(-1,1)$: $1-x^2>0 \implies f'(x)>0$, i.e. the function is increasing.
(c) $(1,\infty)$: $1-x^2<0 \implies f'(x)<0$, i.e. the function is decreasing.

Put this information on a table. Now, we do some analysis.

From (a) to (b), we go past the critical point $x=-1$; the function decreases then increases, making $x=-1$ a **local minimum**.

From (b) to (c), we go past the critical point $x=1$; the function increases then decreases, making $x=1$ a **local maximum**.

Since there is only one local maximum and one local minimum, while knowing that the function goes to the horizontal asymptote at $y=0$, these are guaranteed to be global maximm and minimum respectively.

(6) Find the points of inflection, if any occur, and determine the concavity of the curve.

The suspect points of inflection satisfy

$$0=f''(x)=\frac{2x\left(x^2-3\right)}{\left(1+x^2\right)^3} \implies x=\pm\sqrt{3},0.$$

Note that the numerator is again always positive, so the sign of f'' is completely determined by the sign of $x\left(x^2-3\right)$. They form a partition of the domain as follows:

(a) $\left(-\infty,-\sqrt{3}\right)$: $x<0$ and $x^2-3>0$ so this makes $x\left(x^2-3\right)<0$, i.e. the function is **concaving down**.
(b) $\left(-\sqrt{3},0\right)$: $x<0$ and $x^2-3<0$ so this makes $x\left(x^2-3\right)>0$, i.e. the function is **concaving up**.
(c) $\left(0,\sqrt{3}\right)$:$x>0$ and $x^2-3<0$ so this makes $x\left(x^2-3\right)<0$, i.e. the function is **concaving down**.
(d) $\left(\sqrt{3},\infty\right)$:$x>0$ and $x^2-3>0$ so this makes $x\left(x^2-3\right)>0$, i.e. the function is **concaving up**.

Put this information on a table. Now, we do some analysis.

Now, you observe that through each suspect, the concavity changes. This makes all of them points of inflection.

(7) Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

The table and the plot were shown in class. The key points are the intercepts, critical points and suspect points of inflection.

Problem. (Bonus) Suppose f is a continuous on $[a,b]$ and twice differentiable on (a,b). Further suppose that it has exactly two critical points $\alpha,\beta\in(a,b)$ such that $\alpha<\beta$. Prove that there is some point $c\in(a,b)$ such that $f''(c)=0$.

Proof. Knowing that $f''(x)$ exists everywhere on (a,b) and $f'(\alpha)=f'(\beta)=0$, by the Mean Value Theorem, there exists $c\in(\alpha,\beta)\subset(a,b)$ such that

$$f''(c)=\frac{f'(\beta)-f'(\alpha)}{\beta-\alpha}=0.$$

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