MATH 111-007 QUIZ 8

NOV. 1ST, 2021

Problem 1. Consder $g(x) = \sqrt{4 - x^2}$ on $[-2, 1]$. Find all absolute and local extrema on this interval (recall the steps).

Solution. We follow the strategy outlined in class.

(1) Critical points.
$g'(x) = \frac{-2x}{2\sqrt{4-x^2}} = -\frac{x}{\sqrt{4-x^2}}$. Therefore, the critical points satisfy

$$-\frac{x}{\sqrt{4-x^2}} = 0 \implies x = 0,$$

and also $x = \pm 2$ as they make $g'(x)$ undefined. Note that $x = 2$ is not in $[-2, 1]$. Altogether, we have two critical points

$$x = 0, \quad x = -2.$$

(2) Evaluate the critical points and the endpoints.
We find

$$\begin{aligned} g(0) &= 2 \\ g(-2) &= 0 \\ g(1) &= \sqrt{3} \end{aligned}$$

(3) Compare values.
$g(1)$ is absolute maximum. $g(-2)$ is absolute minimum. $g(0)$ is a local maximum because we look at its neighbouring points such as $g(-1) = \sqrt{3} < g(0) = 2$ and $g(0.5) = \sqrt{3.75} < g(0) = 2$, meaning that $g(0) \geq g(x)$ for all $x \in (-1, 0.5)$. Lastly, $g(1) = \sqrt{3}$ is also a local minimum because locally it loses to all other function values.

Problem 2. We have a piece of square cardboard of side length 2. At each corner, we cut off a square of length x, fold up the sides to make a box with an open top. To maximize the volume of this box, what should x be and what's the resulting maximal volume? (Hint: what's the domain of this problem?)

Solution. The box now has size length $(2 - 2x)$ and a height of x. It has volume

$$V(x) = (2-2x)^2 x = 4(1-x)^2 x = 4\left(x^2 - 2x + 1\right) x = 4x^3 - 8x^2 + 4x.$$

To maximize, we first determine the domain of x, that is, $x \in [0, 1]$.

(1) Critical points.
They satisfy

$$0 = V'(x) = 12x^2 - 16x + 4 \implies 3x^2 - 4x + 1 = 0 \implies (3x-1)(x-1) = 0 \implies x = 1, x = \frac{1}{3}.$$

Your book says that critical points are **interior** points that satisfy zero slope or undefined slope. By this convention, we would say that $x = 1$ is NOT a critical point. But there are also other definitions that do not have this restriction. Anyways, we evaluate $x = 1$ anyways since it's a boundary point.

(2) Evaluate critical points and the endpoints.

$$\begin{aligned} V(0) &= 0 \\ V(1) &= 0 \\ V\left(\frac{1}{3}\right) &= \left(2 - \frac{2}{3}\right)^2 \frac{1}{3} = \frac{16}{9}\frac{1}{3} = \frac{16}{27}. \end{aligned}$$

(3) Compare values.
Certainly $V\left(\frac{1}{3}\right) = \frac{16}{27}$ is the absolute maximum. Since it occurred on the interior, it is also automatically a local maximum.

Problem 3. (Bonus) The height of a body moving vertically (subject to free fall) is given by

$$s = -\frac{1}{2}gt^2 + v_0 t + s_0, \quad g > 0$$

where v_0 and s_0 are the known initial velocity and position respectively. Find the body's maximum height (simplified and expressed in terms of the known constants v_0, s_0 and g).

Solution. Since this is a parabola that opens down, there exists a maximum at $t = t_g$ where $s'(t_g) = 0$. Indeed,

$$s'(t) = -gt + v_0$$

which means

$$0 = s'(t_g) = -gt_g + v_0 \implies t_g = \frac{v_0}{g}.$$

Therefore,

$$s(t_g) = -\frac{1}{2}g\left(\frac{v_0}{g}\right)^2 + v_0\frac{v_0}{g} + s_0 = -\frac{1}{2}\frac{v_0^2}{g} + \frac{v_0^2}{g} + s_0 = \frac{v_0^2}{g} + s_0.$$