MATH 111-007 QUIZ 7

OCTOBER, 25TH, 2021

Problem 1. Find the linearisation of $f(x) = e^{-\frac{x^2}{2}}$ at $x = 0$.

Solution. We follow the definition of linearisation of $f(x)$ at $x = 0$, that is,

$$L_{x=a}(x) = f(a) + f'(a)(x - a)$$

where $a = 0$. We find that $f(0) = 1$ and

$$f'(x) = e^{-\frac{x^2}{2}} \cdot (-x) \implies f'(0) = 0.$$

Therefore,

$$L_{x=0}(x) = 1.$$

A flat line.

Problem 2. Compute dy where $y = f(x) = x^3$ and use differentials to estimate $(2.01)^3$.

Solution.

$$dy = 3x^2 dx.$$

The differential estimate is given by

$$f(x + dx) \approx f(x) + dy.$$

We first note that $(2.01)^3 = (2 + 0.01)^3$. Therefore, with $x = 2$ and $dx = 0.01$, we have

$$\begin{aligned} f(2 + 0.01) &\approx f(2) + dy \\ &= f(2) + f'(2)\,dx \\ &= 8 + 3 \cdot (2)^2 \cdot 0.01 \\ &= 8.12. \end{aligned}$$

True answer is $(2.01)^3 \approx 8.120601$, so our estimate is accurate up to the 4th significant digit. Pretty good.

Problem. (Bonus) Use differentials **or** linearisation to estimate $\sin(1)$. Hint: which (convenient) angle has radian readings close to 1? Leave the expression of your estimate in terms of π.

Solution. The angle close to 1 is $\frac{\pi}{3} \approx 1.04$. Thus, let $f(x) = \sin(x)$, and we estimate via

$$f(x + dx) \approx f(x) + f'(x)\,dx$$

with $x = \frac{\pi}{3}$ and $dx = 1 - \frac{\pi}{3}$. Altogether,

$$\begin{aligned} \sin(1) &= f(1) \\ &= f\left(\frac{\pi}{3} + dx\right) \\ &\approx f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)(dx) \\ &= \sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)\left(1 - \frac{\pi}{3}\right) \\ &= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(1 - \frac{\pi}{3}\right) \\ &\approx 0.84243 \end{aligned}$$

where the exact value is

$$\sin(1) \approx 0.841470.$$

Now we are only accurate up to the second significant digit. This level accuracy (or lack of) is expected because $\theta = \frac{\pi}{3} \approx 1.0472$ which is at a nontrivial distance from where we took the linearization (at $\theta = 1$).