MATH 111-007 QUIZ 6

OCTOBER, 18TH, 2021

Problem 1. Find the derivative of $f(x) = 2^{\ln(\sin(x))}$.

Solution. We note that the outer function is $h(x) = 2^x$ and inner function is $g(x) = \ln(\sin(x))$ so that $f(x) = h(g(x))$ and thus $f'(x) = h'(g(x))\, g'(x)$. We find that

$$h'(x) = 2^x \ln(2) \implies h'(g(x)) = 2^{g(x)} \ln(2) = 2^{\ln(\sin(x))} \ln(2),$$

and

$$g'(x) = \frac{d}{dx} \ln(\sin(x)) = \frac{1}{\sin(x)} \cos(x) \boxed{= \cot(x)}$$

where the boxed step is optional. Altogether,

$$f'(x) = h'(g(x))\, g'(x) = 2^{\ln(\sin(x))} \ln(2) \cot(x).$$

Problem 2. Evaluate $f'\left(\frac{1}{\sqrt{2}}\right)$ where $f(x) = e^{\sin^{-1}(x)}$.

Solution. We identify that the outer function is $h(x) = e^x$ and inner function $g(x) = \sin^{-1}(x)$, which means $f(x) = h(g(x))$ and $f'(x) = h'(g(x)) \cdot g'(x)$. We find

$$h'(x) = e^x \implies h'(g(x)) = e^{\sin^{-1}(x)},$$

and

$$g'(x) = \frac{1}{\sqrt{1 - x^2}}.$$

Altogether,

$$f'(x) = \frac{e^{\sin^{-1}(x)}}{\sqrt{1 - x^2}}.$$

It remains to find out about the function values as we want $f'\left(\frac{1}{\sqrt{2}}\right)$.

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

since $\sin\left(\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right) = \frac{1}{\sqrt{2}}$ (also note that that the range of $\sin^{-1}(x)$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$ is not correct.). Thus,

$$f'\left(\frac{1}{\sqrt{2}}\right) = \frac{e^{\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)}}{\sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2}} = e^{\frac{\pi}{4}} \frac{1}{\sqrt{1 - \frac{1}{2}}} = \sqrt{2} e^{\frac{\pi}{4}}$$

Problem. (Bonus) Do problem 1 in another way, and specify the domain of $f(x)$ and $f'(x)$.

Solution. You do implicit differentiation by first taking ln of both sides.

$$\ln(f(x)) = \ln\left(2^{\ln(\sin(x))}\right) = \ln(\sin(x)) \ln(2).$$

Then we take a derivative of both sides,

$$\frac{1}{f(x)} f'(x) = \ln(2) \frac{d}{dx} \ln(\sin(x)) = \ln(2) \frac{\cos(x)}{\sin(x)} = \ln(2) \cot(x).$$

Thus, multiplying $f(x)$ over to the right hand side, we have

$$f'(x) = \ln(2)\cot(x) f(x) = \ln(2)\cot(x) 2^{\ln(\sin(x))}.$$

For the domain, we first look at $f(x) = 2^{\ln(\sin(x))}$. The only stipulation is that the argument of ln, namely, $\sin(x) > 0$ since the function $\ln(x)$ only allows positive input values. At the same time, $\sin(x) > 0 \implies 0 < x < \pi$ (or in general $2n\pi < x < (2n+1)\pi$ for $n = 0, 1, 2, \ldots$). Thus the domain of $f(x)$ is $0 < x < \pi$.

Now we look at $f'(x) = \ln(2)\cot(x) 2^{\ln(\sin(x))}$. We see the original function in $f'(x)$ so the domain is at most the same as $f(x)$. The extra $\cot(x)$ restricts that $\sin(x) \neq 0$ but $\sin(x) > 0$ anyways, so $\cot(x)$ is fine with the original domain for $f(x)$. Altogether, the domain of $f'(x)$ is $0 < x < \pi$.