MATH 111-007 QUIZ 5

OCTOBER, 11TH, 2021

Problem 1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ where $x^2 + 4y^2 = 1$. (Simplify your final answer, to the best of your ability).

Solution.

$$\begin{aligned} \frac{d}{dx}\left(x^2 + 4y^2\right) &= \frac{d}{dx}(1) \\ \implies 2x + 8y\frac{dy}{dx} &= 0 \\ \implies \frac{dy}{dx} &= -\frac{x}{4y}. \end{aligned}$$

Then,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{x}{4y}\right) = -\frac{1}{4}\frac{d}{dx}\left(\frac{x}{y}\right)$$

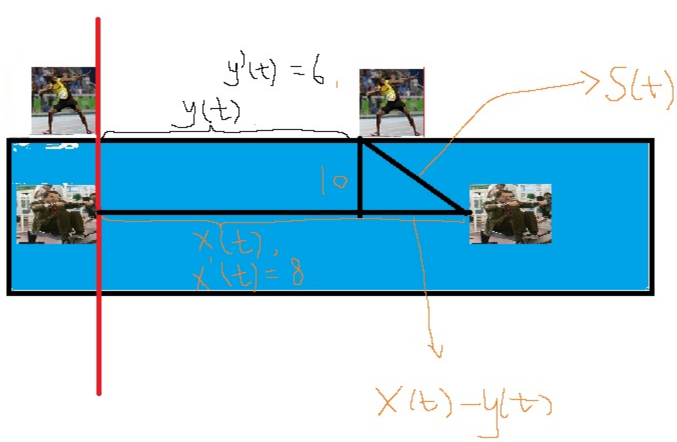
$$= -\frac{1}{4}\frac{y - x\frac{dy}{dx}}{y^2} = \frac{x\frac{dy}{dx} - y}{4y^2} = \frac{x\left(-\frac{x}{4y}\right) - y}{4y^2} = -\frac{\frac{x^2}{4y} + y}{4y^2} = -\frac{x^2 + 4y^2}{16y^3} = -\frac{1}{16y^3}.$$

Problem 2. A runner and a rower are having a race by the straight portion of a river. They begin at the same starting line and go in the same direction (parallel to each other). The runner is running at a pace of 6 meters per second along the shore, while the rower rows at 8 meters per second, in the middle of a still river of 20 meters wide.

Question: find the rate at which the distance between the two is changing 1 minute into the race.

Do **steps 1-4** correctly for full credit and step 5 for a bonus of 2 points.

(1) Sketch;
(2) Introduce and label the variables;
(3) Find the relationship between the variables;
(4) Write down what is known and exactly what the question is looking for in terms of your variables.
(5) Solve (bonus).



Let $x(t)$ and $y(t)$ be the positions of the rower and the runner respectively, at time t. They are always vertically separated by 10 meters. Let $s(t)$ be the distance between the rower and the runner. Then,

$$s(t) = \sqrt{10^2 + (x(t) - y(t))^2}.$$

The question is asking for $\frac{ds}{dt}|_{t=60} = s'(60)$ (convert minute into second).

$$\begin{aligned}\frac{ds}{dt} &= \frac{d}{dt}\sqrt{10^2 + (x(t) - y(t))^2} \\ &= \frac{1}{2}\left(10^2 + (x(t) - y(t))^2\right)^{-\frac{1}{2}} 2(x(t) - y(t))(x'(t) - y'(t)) \\ &= \frac{(x(t) - y(t))(x'(t) - y'(t))}{\sqrt{10^2 + (x(t) - y(t))^2}}\end{aligned}$$

Thus, evaluating at $t = 60$, we have

$$\begin{aligned}\frac{ds}{dt}|_{t=60} &= \frac{(x(60) - y(60))(x'(60) - y'(60))}{\sqrt{10^2 + (x(60) - y(60))^2}} \\ &= \frac{(8 \cdot 60 - 6 \cdot 60)(8 - 6)}{\sqrt{10^2 + (8 \cdot 60 - 6 \cdot 60)^2}} \\ &= \frac{240}{\sqrt{10^2 + 120^2}} \\ &\approx 1.993\end{aligned}$$

though you could estimate this quickly by 2, since $\sqrt{10^2 + 120^2} \approx \sqrt{120^2} = 120$.