MATH 111-007 QUIZ 2

SEPTEMBER 20TH, 2021

Simplify your expressions in the middle of a process (not just at the end of it).

Problem 1. Evaluate the following limits, allowing $+\infty$ and $-\infty$ as possible values of a limit. Explain why if the limit does not exist.

(1) $\lim_{x\to -1} \frac{4x+4}{x^2-2x-3}$.

Solution.

$$\begin{aligned}
\lim_{x\to -1} \frac{4x+4}{x^2-2x-3} &= \lim_{x\to -1} \frac{4(x+1)}{(x-3)(x+1)} \\
&= \lim_{x\to -1} \frac{4}{x-3} \\
&= -1
\end{aligned}$$

(2) $\lim_{x\to 0} \frac{\tan(2x)}{\sin(3x)}$.

Solution.

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan(2x)}{\sin(3x)} &= \lim_{x\to 0} \frac{\sin(2x)}{\cos(2x)} \frac{1}{\sin(3x)} \\
&= \lim_{x\to 0} \left(\frac{\sin(2x)}{2x}\frac{2x}{1}\right)\left(\frac{3x}{\sin(3x)}\frac{1}{3x}\right)\left(\frac{1}{\cos(2x)}\right) \\
&= \lim_{x\to 0} \left(\frac{\sin(2x)}{2x}\right)\left(\frac{3x}{\sin(3x)}\right)\left(\frac{1}{\cos(2x)}\right)\left(\frac{2x}{3x}\right) \\
&= \frac{2}{3}\lim_{x\to 0} \left(\frac{\sin(2x)}{2x}\right)\left(\frac{3x}{\sin(3x)}\right)\left(\frac{1}{\cos(2x)}\right) \\
&= \frac{2}{3}\left(\lim_{x\to 0} \frac{\sin(2x)}{2x}\right)\left(\lim_{x\to 0} \frac{3x}{\sin(3x)}\right)\left(\lim_{x\to 0} \frac{1}{\cos(2x)}\right) \\
&= \frac{2}{3}\cdot 1\cdot 1\cdot 1 \\
&= \frac{2}{3}
\end{aligned}$$

Problem 2. Find the value(s) of a such that the following piecewise function is continuous,

$$f(x) = \begin{cases} 4x, & x < 2, \\ a^2x^2 - 4a, & x \geq 2. \end{cases}$$

Solution. For $x < 2$, we have a line which is continuous. For $x \geq 2$, we have a parabola, which is also continuous. We now consider the point of controversy $x = 2$. In order for $f(x)$ to be continuous, the left and right limit must be equal and equal to the function value (note that the function value $f(2) = 4a^2 - 4a$ is in fact achieved by the second piece since $x \geq 2$ includes $x = 2$, thus making this function so-called "right-continuous" since the function value and the right limit are equal.) By noticing that the right limit and function value are equal, we only now need to force the left limit to be equal to the function value.

$$\lim_{x\to 2^-} f(x) = f(2)$$
$$\implies \lim_{x\to 2^-} 4x = a^2x^2 - 4a$$
$$\implies 8 = 4a^2 - 4a$$
$$\implies a^2 - a - 2 = 0$$
$$\implies (a-2)(a+1) = 0$$
$$\implies a = 2, -1$$

Two values of a grant $f(x)$ continuity.

Problem. (Bonus) Based on problem 2 with the value(s) of a you found, is the function differentiable at $x = 2$? Explain.

Solution. First, note that the derivative of $g(x) = x^2$ is

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0} (2x + h) = 2x$$

and the derivative of a constant is 0.

For $a = 2$, we have

$$f(x) = \begin{cases} 4x, & x < 2 \\ 4x^2 - 8, & x \geq 2 \end{cases}.$$

We check the left and right derivative at $x = 2$. Now, for the left derivative,

$$\lim_{x\to 2^-} f'(x) = \lim_{x\to 2^-} 4 = 4$$

since every line is a tangent line of itself.

$$\begin{aligned} \lim_{x\to 2^+} f'(x) &= \lim_{x\to 2^+} \frac{d}{dx}\left(4x^2 - 8\right) \\ &= \lim_{x\to 2^+} 4\frac{d}{dx}\left(x^2\right) \\ &= \lim_{x\to 2^+} 8x \\ &= 16 \end{aligned}$$

which is different from $\lim_{x\to 2^-} f(x)$. Hence when $a = 2$, $f(x)$ is NOT differentiable at $x = 2$.

For $a = -1$, we have

$$f(x) = \begin{cases} 4x, & x < 2 \\ x^2 + 4, & x \geq 2 \end{cases}.$$

Same procedure. Left derivative at $x = 2$ doesn't change and is $\lim_{x\to 2^-} f'(x) = 4$. Right derivative at $x = 2$ is

$$\lim_{x\to 2^+} f'(x) = \lim_{x\to 2^+} \frac{d}{dx}\left(x^2 + 4\right) = \lim_{x\to 2^+} 2x = 4$$

which is equal to the left derivative $\lim_{x\to 2^-} f'(x)$, thus confirming that the slope at $x = 2$ is exactly 4, granting existence of a derivative at $x = 2$. In other words, when $a = -1$, $f(x)$ IS differentiable at $x = 2$.