MATH 111-007 QUIZ 12

Problem 1. Find

$$\int_{-1}^{1} xe^{-x^2}dx$$

Solution. Several ways.

(1) We make $u = -x^2$ and thus $du = -2xdx$. Thus, we rewrite the original integral in favour of our substitution,

$$\begin{aligned}\int_{-1}^{1} xe^{-x^2}dx &= -\frac{1}{2}\int_{-1}^{1} e^{-x^2}(-2xdx) \\ &\overset{u=-x^2}{=} -\frac{1}{2}\int_{-1}^{-1} e^{u}du \\ &= 0\end{aligned}$$

The last equality uses the fact that $\int_a^a f(x)\,dx = 0$ no matter what a is.

(2) Notice that xe^{-x^2} is an odd function (x is odd, while e^{-x^2} is even; a product of odd and even function gives an odd function). If $f(x)$ is an odd function, it satisfies $f(-x) = -f(x)$. Over a symmetrical interval $[-a, a]$

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &\overset{\text{oddness of } f}{=} \int_{-a}^{a} -f(-x)\,dx \\ &\overset{u=-x}{=} \int_{a}^{-a} f(u)\,du \\ &\overset{\text{properties of def. integral}}{=} -\int_{-a}^{a} f(u)\,du\end{aligned}$$

At the second equality, we substituted $u = -x$ which implies $du = -dx$. At the same time, the integrating limits swap

$$\begin{aligned}&\text{(upper limit)} \quad x = a \implies u = -a \\ &\text{(lower limit)} \quad x = -a \implies u = a\end{aligned}$$

Now, u is a dummy variable, so we can replace x back. The above informs us that

$$\int_{-a}^{a} f(x)\,dx = -\int_{-a}^{a} f(x)\,dx$$

which implies that both sides must be 0 (i.e. if we let $b = \int_{-a}^{a} f(x)\,dx$, then we have $b = -b$, where the only solution is $b = 0$). Thus, a definite integral over a symmetric interval of an odd function is 0.

Problem 2. Evaluate

$$\int \sec^2(x)\tan(x)\,dx$$

Solution. As we pointed out in class, there are two ways to do this, resulting in an equivalent answer.

(1) Make $u = \tan(x)$ and thus $du = \sec^2(x)\,dx$. We have

$$\begin{aligned}\int \sec^2(x)\tan(x)\,dx &= \int \underbrace{\tan(x)}_{u}\underbrace{\sec^2(x)\,dx}_{du} \\ &\overset{u=\tan(x)}{=} \int u du \\ &= \frac{u^2}{2} + C \\ &= \frac{\tan^2(x)}{2} + C\end{aligned}$$

(2) A little detour. Make $u = \sec(x)$ and $du = \sec(x)\tan(x)\,dx$. We have

$$\begin{aligned}\int \sec^2(x)\tan(x)\,dx &= \int \underbrace{\sec(x)}_{u}\underbrace{\sec(x)\tan(x)\,dx}_{du} \\ &\overset{u=\sec(x)}{=} \int u du \\ &= \frac{u^2}{2} + D \\ &= \frac{\sec^2(x)}{2} + D\end{aligned}$$

Why are both of these the right answer? How are they equivalent? Note that C and D are **any** constants. So, we can massage it. Take the first integral, the final answer is

$$\begin{aligned}\frac{\tan^2(x)}{2} + C &= \frac{\tan^2(x)}{2} + \frac{1}{2} + C - \frac{1}{2} \\ &= \frac{\tan^2(x)+1}{2} + C - \frac{1}{2} \\ &= \frac{\sec^2(x)+1}{2} + C - \frac{1}{2}\end{aligned}$$

Since C and D are **any constants**, it hurts no one to write $D = C - \frac{1}{2}$. Therefore, both answers are equivalent and correct.

Problem 3. Evaluate

$$\int_0^{\pi^2} \frac{\cos(\sqrt{x})\,dx}{\sqrt{x}}$$

Solution. We make $u = \sqrt{x}$ as it deals with the most complicated part of the problem. We find $du = \frac{1}{2\sqrt{x}}dx$. The integrating bounds change as follows

$$\begin{aligned}x = 0 &\implies u = \sqrt{0} = 0 \\ x = \pi^2 &\implies u = \sqrt{\pi^2} = \pi\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^{\pi^2} \frac{\cos(\sqrt{x})\,dx}{\sqrt{x}} &= 2\int_0^{\pi^2} \cos\left(\sqrt{x}\right)\left(\frac{1}{2\sqrt{x}}dx\right) \\ &\overset{u=\sqrt{x}}{=} 2\int_0^{\pi} \cos(u)\,du \\ &= 2\left[\sin(u)\right]_{x=0}^{x=\pi} \\ &= 2\left[\sin(\pi) - \sin(0)\right] \\ &= 0\end{aligned}$$

Problem. (Bonus)

$$\int \cos^3(x)\,dx$$

Solution. We chop into things we know.

$$\begin{aligned}
\int \cos^3(x)\,dx &= \int \cos^2(x)\cos(x)\,dx \\
&= \int \left(1-\sin^2(x)\right)\cos(x)\,dx \\
&\overset{u=\sin(x)}{=} \int \left(1-u^2\right)du \\
&= u - \frac{u^3}{3} + C \\
&= \sin(x) - \frac{\sin^3(x)}{3} + C
\end{aligned}$$