MATH 111-007 QUIZ 11

Problem 1. Evaluate the following integrals.

(1) $\int \left(\frac{1}{3}\sin(3x) + \frac{1}{2}\cos(2x)\right) dx$

Solution. We utilise the linearity of integration, and then divide and conquer,

$$\begin{aligned}\int \left(\frac{1}{3}\sin(3x) + \frac{1}{2}\cos(2x)\right) dx &= \frac{1}{3}\int \sin(3x)\,dx + \frac{1}{2}\int \cos(2x)\,dx \\ &= \frac{1}{3}\left(-\frac{1}{3}\cos(3x)\right) + \frac{1}{2}\left(\frac{1}{2}\sin(2x)\right) + C \\ &= -\frac{1}{9}\cos(3x) + \frac{1}{4}\sin(2x) + C\end{aligned}$$

(2) $\int \left(x^4 + 3x + e^{-\frac{x}{2}}\right) dx$

Solution. We check term by term,

$$\begin{aligned}\int \left(x^4 + 3x + e^{-\frac{x}{2}}\right) dx &= \int x^4 dx + 3\int x dx + \int e^{-\frac{x}{2}} dx \\ &= \frac{x^5}{5} + 3\left(\frac{x^2}{2}\right) + (-2)\, e^{-\frac{x}{2}} + C \\ &= \frac{x^5}{5} + \frac{3}{2}x^2 - 2e^{-\frac{x}{2}} + C\end{aligned}$$

Problem 2. Estimate the area under the graph of $f(x) = \sin(2x)$ with 4 subintervals on $[0, \pi]$, using either left or right endpoint rule. (Hint: write out what the partition is and do a simple sketch.)

Solution. The partition is

$$P = \left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\right\}$$

and we find that each subinterval has length

$$\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$$

(1) For left endpoint rule, we consider the first four points, that is,

$$\begin{aligned}Area &= \Delta x \left(f(0) + f\left(\frac{\pi}{4}\right) + f\left(\frac{3\pi}{4}\right) + f(\pi)\right) \\ &= \frac{\pi}{4}\left(\sin(0) + \sin\left(2\cdot\frac{\pi}{4}\right) + \sin\left(2\cdot\frac{\pi}{2}\right) + \sin\left(2\cdot\frac{3\pi}{4}\right)\right) \\ &= \frac{\pi}{4}(0 + 1 + 0 + (-1)) \\ &= 0\end{aligned}$$

(2) For right endpoint rule, we consider the last four points, that is,

$$\begin{aligned}Area &= \Delta x \left(f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right) + f\left(\frac{3\pi}{4}\right) + f(\pi)\right) \\ &= \frac{\pi}{4}\left(\sin\left(2\cdot\frac{\pi}{4}\right) + \sin\left(2\cdot\frac{\pi}{2}\right) + \sin\left(2\cdot\frac{3\pi}{4}\right) + \sin(2\cdot\pi)\right) \\ &= \frac{\pi}{4}(1 + 0 + (-1) + 0) \\ &= 0\end{aligned}$$

Problem. (Bonus) Use midpoint rule for problem 2 with a legible sketch. Does the answer make sense? With the sketch, could you guess what the actual integral is, i.e. $\int_0^\pi \sin(2x)\,dx = ?$

Solution. We have the same partition. But it is also helpful to write down what the midpoints are. Given

$$P = \left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\right\},$$

we have midpoints

$$\mathcal{M} = \left\{\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}\right\}.$$

Therefore,

$$\begin{aligned} Area &= \Delta x \left(f\left(\frac{\pi}{8}\right) + f\left(\frac{3\pi}{8}\right) + f\left(\frac{5\pi}{8}\right) + f\left(\frac{7\pi}{8}\right)\right) \\ &= \frac{\pi}{4}\left(\sin\left(2\cdot\frac{\pi}{8}\right) + \sin\left(2\cdot\frac{3\pi}{8}\right) + f\left(2\cdot\frac{5\pi}{8}\right) + f\left(2\cdot\frac{7\pi}{8}\right)\right) \\ &= \frac{\pi}{4}\left(\sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{3\pi}{4}\right) + f\left(\frac{5\pi}{4}\right) + f\left(\frac{7\pi}{4}\right)\right) \\ &= \frac{\pi}{4}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{2}}{2}\right)\right) \\ &= 0 \end{aligned}$$

With a legible sketch, you will see that the area on the left of $x = \frac{\pi}{2}$ is equal in magnitude to that on the right, just in different signs. Therefore, the two areas cancel and we should anticipate that $\int_0^\pi \sin(2x)\,dx = 0$.