MATH 111-007 QUIZ 10

NOVEMBER 15TH, 2021

Instructions: there are a total of **three** problems and one bonus.

Problem 1. (2 points a piece) Evaluate the following limits.

(1) $\lim_{x\to 1} \frac{5x^4-4x^2-1}{10-x-9x^3}$.

Solution.

$$\begin{aligned} \lim_{x\to 1} \frac{5x^4-4x^2-1}{10-x-9x^3} &\overset{\text{"}\frac{0}{0}\text{", L'H.}}{=} \lim_{x\to 1} \frac{\frac{d}{dx}\left(5x^4-4x^2-1\right)}{\frac{d}{dx}\left(10-x-9x^3\right)} \\ &= \lim_{x\to 1} \frac{20x^3-8x}{-1-27x^2} \\ &= -\frac{12}{28} \\ &= -\frac{3}{7}. \end{aligned}$$

(2) $\lim_{\theta\to\frac{\pi}{2}} \frac{\sin\theta-1}{\theta-\pi}$.

Solution.

$$\lim_{\theta\to\frac{\pi}{2}} \frac{\sin\theta-1}{\theta-\pi} = \frac{\sin\left(\frac{\pi}{2}\right)-1}{\frac{\pi}{2}-\pi} = \frac{1-1}{-\frac{\pi}{2}} = 0.$$

Problem 2. (3 points) A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides. Given 100 ft of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?

Solution. Let x and y be the length and width of the garden respectively. Then, the area satisfies

$$A = xy.$$

At the same time, we are also subject to the constraint

$$x + 2y = 100 \implies x = 100 - 2y.$$

From this, we deduce that physically we must have the range of values

$$0 \le y \le 50.$$

Altogether, we now attempt to maximize

$$A(y) = (100-2y)\,y = 100y - 2y^2.$$

The critical points satisfy

$$0 = A'(y) = 100 - 4y \implies y = 25.$$

The endpoints evaluate to

$$A(0) = 0, \quad A(50) = 0$$

while

$$A(25) = 1250 > 0.$$

This automatically guarantees it is a global maximum (global extrema happen at the endpoints or at critical points).

Problem 3. (3 points) Use Newton's method to find $4^{\frac{1}{3}}$ by estimating the zeros of $f(x) = x^3 - 4$. Start with $x_0 = 1$ and find x_2, that is, iterate the Newton's method formula twice.

Solution. We recall that Newton's method has formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

When $f(x) = x^3 - 4$ and thus $f'(x) = 3x^2$, we simplify the formula to

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n^3 - 4}{3x_n^2} \\ &= x_n - \frac{x_n^3}{3x_n^2} + \frac{4}{3x_n^2} \\ &= x_n - \frac{1}{3}x_n + \frac{4}{3x_n^2} \\ &= \frac{2}{3}x_n + \frac{4}{3x_n^2} \end{aligned}$$

Now, with $x_0 = 1$, we have (by directly plugging in)

$$x_1 = \frac{2}{3}x_0 + \frac{4}{3x_0^2} = \frac{2}{3} \cdot 1 + \frac{4}{3 \cdot 1^2} = \frac{2}{3} + \frac{4}{3} = 2,$$

and

$$x_2 = \frac{2}{3}x_1 + \frac{4}{3x_1^2} = \frac{2}{3} \cdot 2 + \frac{4}{3 \cdot 2^2} = \frac{4}{3} + \frac{1}{3} = \frac{5}{3}.$$

Problem. (Bonus, 2 points) Consider the taxicab distance between the seats in the current classroom. In other words, seats one unit distance away from you only include those that are exactly one seat to your left, right, up and down. Write down the names of the students who are sitting less or equal to **two units** away from you. (Extra 1 point if you can present the names visually to showcase the distance)

Check out Taxicab Distance.