RECITATION 0930

Problem 1. $\int \frac{dx}{\sqrt{36+x^2}}$

Solution. Let $x = 6\tan\theta$ and therefore $dx = 6\sec^2\theta d\theta$.

$$\begin{aligned}\int \frac{dx}{\sqrt{36+x^2}} &= \int \frac{6\sec^2\theta d\theta}{\sqrt{36+36\tan^2\theta}} \\ &= \int \frac{6\sec^2\theta d\theta}{6\sec\theta} \\ &= \int \sec\theta d\theta \\ &\overset{\text{(why?)}}{=} \ln|\tan\theta + \sec\theta| + C\end{aligned}$$

Solving θ for x using the original substitution yields,

$$\theta = \tan^{-1}\left(\frac{x}{6}\right).$$

Drawing a right triangle with an opposite of x and adjacent of 6, we find

$$\tan\theta = \frac{x}{6}, \quad \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{6}{\sqrt{x^2+36}}} = \frac{\sqrt{x^2+36}}{6}$$

and altogether,

$$\int \frac{dx}{\sqrt{36+x^2}} = \ln\left|\frac{x}{6} + \frac{\sqrt{x^2+36}}{6}\right| + C$$

Problem 2. $\int \frac{x^2 dx}{\sqrt{9-16x^2}}$

Solution. Let $x = \frac{3}{4}\sin\theta$ (why not $\cos\theta$? Does it matter?) and thus $dx = \frac{3}{4}\cos\theta d\theta$.

$$\begin{aligned}\int \frac{x^2 dx}{\sqrt{9-16x^2}} &= \int \frac{\frac{9}{16}\sin^2\theta \frac{3}{4}\cos\theta d\theta}{\sqrt{9-9\sin^2\theta}} \\ &= \frac{9}{64}\int \frac{\sin^2\theta\cos\theta d\theta}{\cos\theta} \\ &= \frac{9}{64}\int \sin^2\theta d\theta \\ &= \frac{9}{128}\int (1-\cos 2\theta)\, d\theta \\ &= \frac{9}{128}\left(\theta - \frac{\sin(2\theta)}{2}\right) + C \\ &= \frac{9}{128}(\theta - \sin\theta\cos\theta) + C\end{aligned}$$

Solving θ for x using the original substitution, we have (draw the triangle yourself)

$$\theta = \sin^{-1}\left(\frac{4x}{3}\right), \quad \sin\theta = \frac{4x}{3}, \quad \cos\theta = \frac{\sqrt{9-16x^2}}{3}$$

and altogether (after algebraic rearrangements)

$$\int \frac{x^2 dx}{\sqrt{9-16x^2}} = \frac{1}{128}\left(9\sin^{-1}\left(\frac{4x}{3}\right) - 4x\sqrt{9-16x^2}\right) + C$$

Problem 3. $\int \frac{(x^2-4)^{3/2}}{x} dx$

Solution. Let $x = 2\sec\theta$ and therefore $dx = 2\sec\theta\tan\theta d\theta$.

$$\begin{aligned}\int \frac{(x^2-4)^{3/2}}{x}dx &= \int \frac{(4\sec^2\theta - 4)^{3/2}}{2\sec\theta} 2\sec\theta\tan\theta d\theta \\ &= \int (4\tan^2\theta)^{3/2}\tan\theta d\theta \\ &= 8\int \tan^4\theta d\theta \\ &= 8\int \tan^2\theta\left(\sec^2\theta - 1\right)d\theta \\ &= 8\left(\int \tan^2\theta\sec^2\theta d\theta - \int\tan^2\theta d\theta\right) \\ &\overset{u=\tan\theta}{=} 8\left(\int u^2 du - \int\left(\sec^2\theta - 1\right)d\theta\right) \\ &= \frac{8\tan^3\theta}{3} - 8\tan\theta + 8\theta + C\end{aligned}$$

(as long as you get here, you should get at least 7/10 for the problem). Solving θ for x using the original substitution, we have (draw the triangle yourself)

$$\theta = \sec^{-1}\left(\frac{x}{2}\right), \quad \tan\theta = \frac{\sqrt{x^2-4}}{2}$$

and altogether

$$\int \frac{(x^2-4)^{3/2}}{x}dx = \boxed{\frac{(x^2-4)^{\frac{3}{2}}}{3} - 4\sqrt{x^2-4} + 8\sec^{-1}\left(\frac{x}{2}\right) + C}.$$

There may be various other equivalent answers due to different ways of factoring or algebraic rearrangement. One possible answer is by reducing the answer in θ to

$$\begin{aligned}\frac{8\tan^3\theta}{3} - 8\tan\theta + 8\theta + C &= \frac{8}{3}\tan\theta\left(\tan^2\theta - 3\right) + 8\theta + C \\ &= \frac{8}{3}\tan\theta\left(\sec^2\theta - 4\right) + 8\theta + C\end{aligned}$$

and thus the substitution will yield

$$\frac{4}{3}\sqrt{x^2-4}\left(\frac{x^2}{4} - 4\right) + 8\sec^{-1}\left(\frac{x}{2}\right) + C = \boxed{\frac{1}{3}\sqrt{x^2-4}\left(x^2-16\right) + 8\sec^{-1}\left(\frac{x}{2}\right) + C}$$