Problem 1. Find the arc length of the graph $y = \int_0^x \sqrt{1 - 2\cos(2z)}dz$ on the interval $\left[0, \frac{\pi}{2}\right]$.

Solution. Note that the given y is still a function $f(x)$ where the independent variable x is in the integrating upper bound. This screams for something relating to the Fundamental Theorem of Calculus. All we need is

$$\frac{dy}{dx} = \sqrt{1 - 2\cos(2x)}$$

and thus with the use of double angle formula (at the 4th equality sign below),

$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \int_0^{\frac{\pi}{2}} \sqrt{1 + (1 - 2\cos(2x))}dx \\ &= \int_0^{\frac{\pi}{2}} \sqrt{2 - 2\cos(2x)}dx \\ &= \int_0^{\frac{\pi}{2}} \sqrt{2 - 2\left(1 - 2\sin^2(x)\right)}dx \\ &= 2\int_0^{\frac{\pi}{2}} \sin(x)\, dx \\ &= 2\left[-\cos(x)\right]_0^{\frac{\pi}{2}} \\ &= 2 \end{aligned}$$

Problem 2. Find the are of the surface generated by revolving the curve $y = \sqrt{3 - x^2}$ about the x-axis on the interval $\left[\sqrt{3}, \sqrt{12}\right]$.

Solution. If you carefully draw the graph represented by the function, you will find a semicircle of radius $\sqrt{3}$ that lies about the x-axis. Then you will find that the interval lies completely outside the domain of the function. Your answer is automatically 0, since you are rotating, NOTHING.

Problem 3. Consider a 5-lb bucket containing 10-lb of water, hanging at the end of a 30-ft rope with weight density 0.5 lb/ft. The rope is pulled up at a rate of 3 ft/s, while the water leaks at a rate of 0.25 lb/s. Find the total work of pulling the rope and the bucket of water up to a roof at 40-ft high.

Solution. Three parts. Let x be the distance from the roof to the bucket (and thus the end of the rope). This tells me that $0 \leq x \leq 30$ where $x = 0$ means at the roof, and $x = 30$ means at the tip of the rope at the beginning.

(1) Bucket only

Easiest part since its weight doesn't change.

$$W_{bucket} = 5\,lb \times 30\,ft = 150\,lb \cdot ft$$

(2) Rope

Harder part since the rope's weight changes as you pull. We consider first the infinitesimal work needed to pull the rope up by dx at arbitrary position $x\,ft$ from the roof. This part of the rope weights $0.5dx\,lb$ and thus (differential) work needed to lift this part is

$$dW_{rope} = 0.5xdx\,ft \cdot lb$$

Then, we simply add up all positions x and find

$$W_{rope} = \int_0^{30} dW_{rope} = \int_0^{30} 0.5xdx = 225\,ft \cdot lb$$

(3) Water

Hardest part of the problem as it involves time and space. However, the principal ingredient doesn't change. We want

$$W_{water} = \int_0^{30} [weight\,of\,water](x)\,dx$$

At the moment, we don't know the weight of water as a function of position x, but time t.

$$weight\,of\,water\,(t) = 10 - \frac{t}{4}$$

which we realise that the water won't be emptied by the role is fully pulled up (note it takes 10 seconds to pull up, while for $t = 10$ here, water still weights $7.5\,lb$). We will arrive at the following weird integral,

$$W_{water} = \int_{x=0}^{x=30} \left(10 - \frac{t}{4}\right) dx$$

It looks like the function we integrate over is solely dependent on t yet the integrating variable is in dx. What we need now is a relationship between x and t, which is given by the rate of pulling of the rope.

$$x = 30 - 3t$$

since at time $t = 0$, the water is 30 ft below the roof, and it is being lifted at a rate $-3\,ft/s$, and hence the slope here. This is our change of variable/substitution, if you will.

$$dx = -3dt$$

and note that the bounds ought to change as well,

$$x = 30 \implies t = 0$$
$$x = 0 \implies t = 10$$

Therefore,

$$W_{water} = \int_{x=0}^{x=30} \left(10 - \frac{t}{4}\right) dx = \int_{t=10}^{t=0} \left(10 - \frac{t}{4}\right) (-3)\,dt = 262.5\,lb \cdot ft$$

Altogether,

$$W = W_{bucket} + W_{rope} + W_{water} = 637.5\,lb \cdot ft$$