Project Notes: Adam Yanco

Project Title: The Number of Maximal Independent Sets in Higher Dimensional Tori Graphs
Name:

Note Well: There are NO SHORT-cuts to reading journal articles and taking notes from them. Comprehension is paramount. You will most likely need to read it several times, so set aside enough time in your schedule.

Contents:

Knowledge Gaps:

This list provides a brief overview of the major knowledge gaps for this project, how they were resolved and where to find the information.

Knowledge Gap	Resolved By	Information is located	Date resolved
What families of graphs are possible/fast to analyze/interesting?			
Program efficiency when counting MIS			

Literature Search Parameters:

These searches were performed between (Start Date of reading) and XX/XX/2019.
List of keywords and databases used during this project.

<table>
<tr><th colspan="2">Database/search engine</th><th>Keywords</th><th>Summary of search</th></tr>
<tr><td colspan="2">Google</td><td>Maximal Independant Sets</td><td></td></tr>
<tr><td>Google</td><td>Maximum Independant Sets</td><td colspan="2">https://projecteuclid.org/journals/taiwanese-journal-of-mathematics/volume-4/issue-4/THE-NUMBER-OF-MAXIMUM-INDEPENDENT-SETS-IN-GRAPHS/10.11650/twjm/1500407302.pdf</td></tr>
<tr><td colspan="2">Google</td><td>Fibonacci number of graphs</td><td>Learned background information following reading source #4</td></tr>
</table>

Tags:

Tag Name	
MIS	Existence
Counting	Proof
Independent	Fibonacci

Template

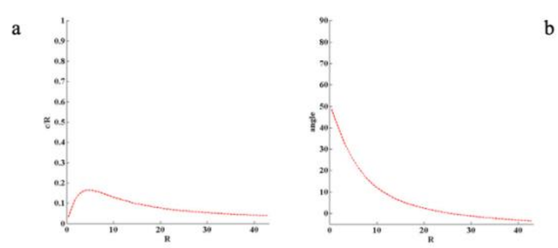
Article notes should be on separate sheets

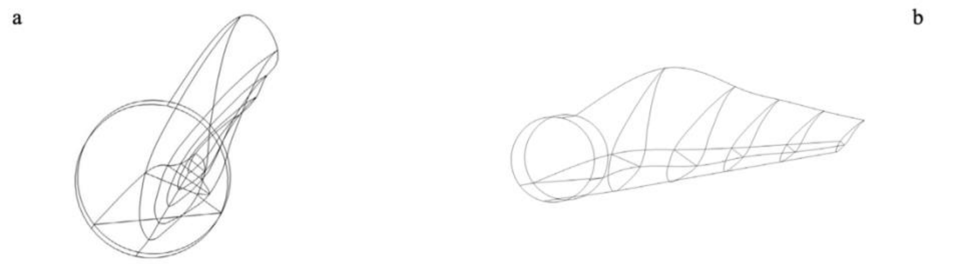
KEEP THIS BLANK AND USE AS A TEMPLATE

Source Title	
Source citation (APA Format)	
Original URL	
Source type	
Keywords	
#Tags	
Summary of key points + notes (include methodology)	
Research Question/Problem/ Need	
Important Figures	
VOCAB: (w/definition)	
Cited references to follow up on	
Follow up Questions	

Article #1 Notes: Aerodynamics and structural analysis of wind turbine blade

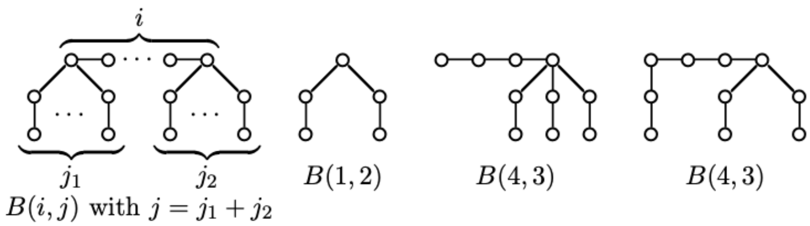
Article notes should be on separate sheets

Source Title	Aerodynamics and structural analysis of wind turbine blade
Source citation (APA Format)	Mouhsine, S. E., Oukassou, K., Ichenial, M. M., Kharbouch, B., & Hajraoui, A. (2018). Aerodynamics and structural analysis of wind turbine blade. *Procedia Manufacturing, 22*, 747–756. https://doi.org/10.1016/j.promfg.2018.03.107
Original URL	https://www.sciencedirect.com/science/article/pii/S2351978918304037?via%3Dihub
Source type	Article on Manufacturing
Keywords	Pitch, Aerodynamics, Torque, Airfoil,
#Tags	#Design #Aerodynamics #Airfoil
Summary of key points + notes (include methodology)	Wind turbines use airfoils to transfer the wind's kinetic energy into usable rotational energy, and thus the efficiency of the airfoils is one of the most important considerations when designing a windmill. Designing an efficient airfoil is difficult, with many considerations, including structural integrity, control problems, and aerodynamics. Using complicated fluid-dynamics equations, the most efficient design (for aerodynamics) can be created, "consisting of airfoils sections of increasing width, thickness and twist angle towards the hub".
Research Question/Problem/ Need	What's the optimal airfoil shape and pitch to maximize a windmill's wind-to-torque conversion efficiency?
Important Figures	 Fig. 5. (a) Chord length distribution as function of radius, (b) twist distribution as function of radius for Horizontal-Axis wind turbine calculated with blade element momentum methods.

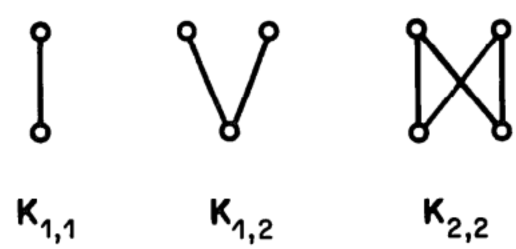
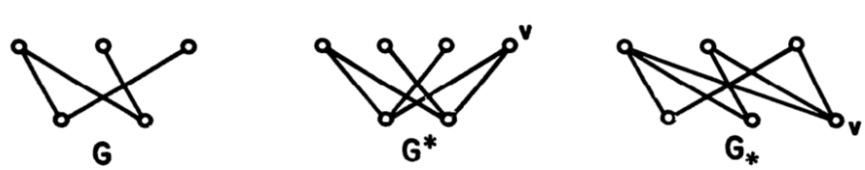
	 Fig. 1. (a) Angle of Attack and Chord Line of an Airfoil, (b) Airfoil cross-sections used in the design of the wind turbine rotor blades
VOCAB: (w/definition)	Torque - a twisting force that causes rotation. Airfoil - a structure with curved surfaces designed to give the most favorable ratio of lift to drag in flight
Cited references to follow up on	- E. Benini, Ã. Significance of blade element theory in performance prediction of marine propellers, Ocean Eng, 31, 957–974, 2004. - M.O.L. Hansen, Aerodynamics of Wind Turbines: Rotors, Loads and Structure, James & James Science Publishers, London, UK, pp. 48-59, 2000. - P. Fuglsang, H.A. Madsen, Optimization method for wind turbine rotors,J. Wind Eng,Ind.Aerodyn. 80 (1999) 191–206.
Follow up Questions	1. Are the current paradigms (separate blades rotating around a single shaft) necessarily the best choices to make, or are there completely different shapes or methods of energy conversion that might be better, but excluded in their search? 2. Is the optimal pitch angle (and maybe shape too) dependent on the wind conditions in which the blade is to be placed, or is the optimal shape the same for all speeds? 3. Are there less efficient designs that end up being more practical (maybe they cause less wear & tear or hold up better)?

Article #2 Notes: The Number of Maximum Independent Sets in Graphs

Source Title	The Number of Maximum Independent Sets in Graphs
Source citation (APA Format)	Jou, M., & Chang, G. (2000). The Number of Maximum Independent Sets in Graphs. *Taiwanese Journal of Mathematics*, *4*(4), 685–695. Project Euclid. https://projecteuclid.org/journals/taiwanese-journal-of-mathematics/volume-4/issue-4/THE-NUMBER-OF-MAXIMUM-INDEPENDENT-SETS-IN-GRAPHS/10.11650/twjm/1500407302.pdf
Original URL	https://projecteuclid.org/journals/taiwanese-journal-of-mathematics/volume-4/issue-4/THE-NUMBER-OF-MAXIMUM-INDEPENDENT-SETS-IN-GRAPHS/10.11650/twjm/1500407302.pdf
Source type	Journal Article
Keywords	Maximal, Maximum, Independent, Graph, Subset
#Tags	#MIS #Counting
Summary of key points + notes (include methodology)	This source counts the number of maximum independent sets. Previous papers have found the size of the maximum, or the number of maximal independent sets (which I'm also planning to look into). Uses some elementary lemmas (such as xi(G) ≤ mi(G)) to get formulas bounding mi(G)
Research Question/Problem/ Need	How many sets of maximum size are there for different types of graphs?

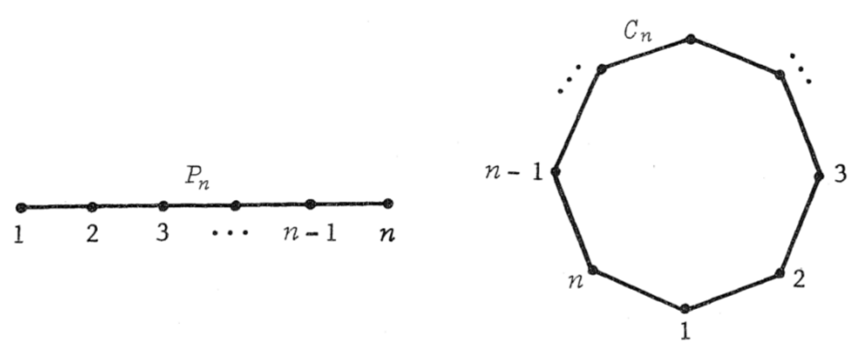
Important Figures	 Figure 1. Batons.  Figure 2. The graph $T'(n)$.
VOCAB: (w/definition)	Baton - graphs obtained from a path P of i ≥ 1 vertices by attaching j ≥ 0 paths of length two to the endpoints of P in all possible ways
Cited references to follow up on	H. S. Wilf, The number of maximal independent sets in a tree, SIAM J. Alge- braic Discrete Methods 7 (1986), 125-130. J. Zito, The structure and maximum number of maximum independent sets in trees, J. Graph Theory 15 (1991), 207-221. V. Linek, Bipartite graphs can have any number of independent sets, Discrete Math. 76 (1989), 131-136.
Follow up Questions	A lot of their formulas just express bounds, but are there ways to use properties of the graphs to get explicit expressions for the number of Maximum Independent Sets? Are explicit formulas difficult to find because no inductive methods can be used to get recursive formulas for graphs such as batons? Is it easier to count maximal or maximum independent sets? Does it depend on the graph? How do the number of each relate to one another?

Article #3 Notes: Bipartite graphs can have any number of independent sets

Source Title	Bipartite graphs can have any number of independent sets
Source citation (APA Format)	Linek, V. (1989). Bipartite graphs can have any number of independent sets. *Discrete Mathematics*, *76*(2), 131–136. https://doi.org/10.1016/0012-365x(89)90306-3
Original URL	https://sciencedirect.com/science/article/pii/0012365X89903063
Source type	Journal Article
Keywords	Bipartite, Subgraph, Induction, Connected
#Tags	#Bipartite, #Existence, #Proof, #Independent
Summary of key points + notes (include methodology)	The paper uses inductive arguments to present a proof. Some cases are eliminated (For example, any number even number of maximal independent sets can be created by adjoining K_1 to a graph with half as many MIS. For a similar reason, only PRIME numbers need to be shown to be possible, but it’s proven in the paper for all odd numbers. While this doesn’t focus specifically on *Maximal* Independent Sets, learning about the properties of Independent Sets still seems important for my project.
Research Question/Problem/ Need	Is there a bipartite graph with N independent sets for every positive integer N?
Important Figures	  **Fig. 1. Upper and lower extensions of $G \in \mathcal{G}_{2,3}$.**

VOCAB: (w/definition)	Bipartite - a graph whose vertices can be divided into two disjoint sets Upper Extension – the maximum number of edges for a collection of vertices that would cause the graph to satisfy a certain property (In this case MIS) Lower Extension – the minimum number of edges for a collection of vertices that would cause the graph to satisfy a certain property (In this case MIS)
Cited references to follow up on	H. Prodinger and R.F. Tichy, Fibonacci numbers of graphs, Fibonacci Quarterly 20 (1982) 16-21.
Follow up Questions	Are there bipartite graphs with N MAXIMAL independent sets for every positive integer N? Are there COMPLETE bipartite graphs with N independent sets for every positive integer N? Are there COMPLETE bipartite graphs with N MAXIMAL independent sets for every positive integer N?

Article #4 Notes: Fibonacci Numbers of Graphs

Source Title	Fibonacci Numbers of Graphs
Source citation (APA Format)	Prodinger, H., & Tichy, R. F. (1979). Fibonacci numbers of graphs. *The Fibonacci Quarterly*, *20*(1), 16–21. https://doi.org/10.1080/00150517.1982.12430021
Original URL	https://www.fq.math.ca/Scanned/20-1/prodinger.pdf
Source type	Journal Article
Keywords	Fibonacci number, Lucas number, Lattice Graph
#Tags	#Independent, #Fibonacci, #Counting
Summary of key points + notes (include methodology)	The number of independent sets of a cycle graph of N elements is the Nth lucas number. The number of independent sets of a line graph of N elements is the N-1th fibonacci number. It defined the 'Fibonacci number of a graph' as the number of Independent sets on it. This shows a deeper structure between adding one node to line or cycle graphs and combining independent sets in recursive way. This looks promising for my project, as I was hoping to prove an inductive property about Independent Sets on tori graphs, which are really just the cartesian product of a large number of cycle graphs.
Research Question/Problem/ Need	How can the counting of independent sets be related to other mathematical structures?
Important Figures	 Fig. 1

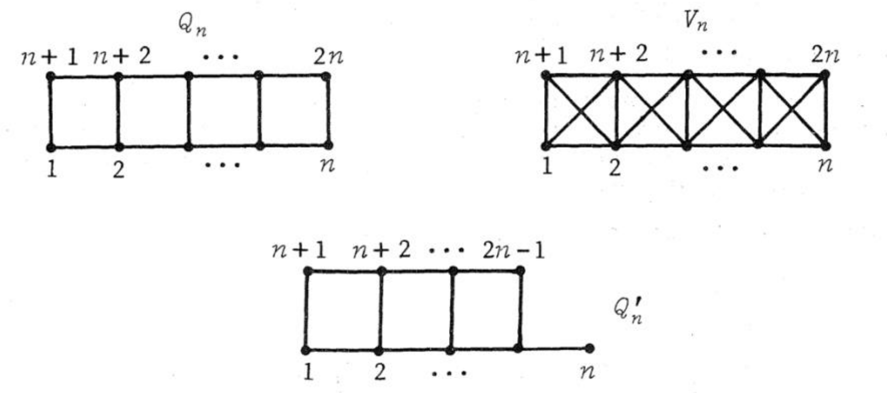
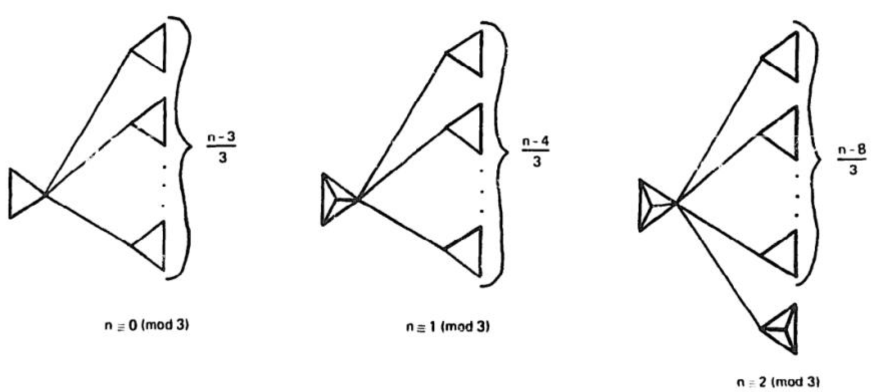
<table>
<tr><td></td><td>

Fig. 5</td></tr>
<tr><td>VOCAB: (w/definition)</td><td>Fibonacci number (of a graph): The number of Independant Sets of the graph</td></tr>
<tr><td>Cited references to follow up on</td><td>L. Comtet. Advanced Combinatorics. Dordrecht-Holland: D. Reidel, 1974.
F. Harary. Graph Theory. Reading, Mass.: Addison-Wesley, 1972.</td></tr>
<tr><td>Follow up Questions</td><td>Why is there a link between Fibonacci numbers and Independent Sets?

Can this recursive property be applied to Maximal Independent Sets?

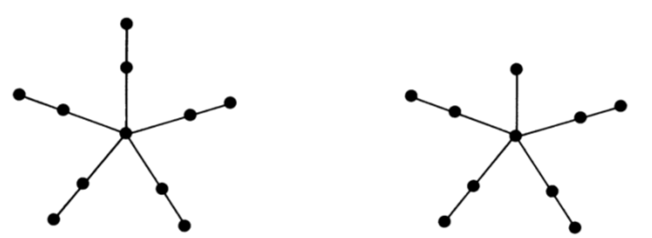
Can this recursive property be applied to the cartesian product of cycle and/or line graphs?</td></tr>
</table>

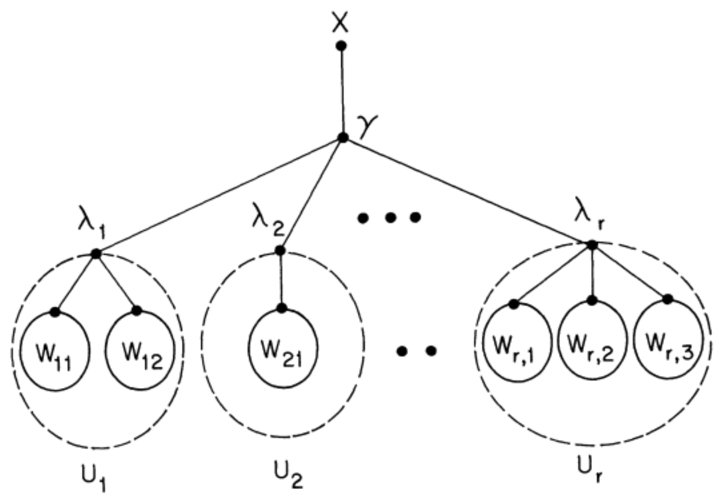
Article #5 Notes: The number of maximal independent sets in connected graphs

Source Title	The number of maximal independent sets in connected graphs
Source citation (APA Format)	Griggs, J. R., Grinstead, C. M., & Guichard, D. R. (1988). The number of maximal independent sets in a connected graph. *Discrete Mathematics*, *68*(2–3), 211–220. https://doi.org/10.1016/0012-365x(88)90114-8
Original URL	https://www.sciencedirect.com/science/article/pii/0012365X88901148?via%3Dihub
Source type	Journal Article
Keywords	Clique
#Tags	#MIS, #Counting, #Cliques
Summary of key points + notes (include methodology)	The paper presents combinatorial techniques for deriving formulas to count the number of maximal independent sets based on graph properties such as vertex connectivity and size. The findings indicate that the number of maximal independent sets can vary dramatically across different graph structures, which could have implications for applications in network theory and optimization.
Research Question/Problem/ Need	In connected graphs, what is the most MIS that a graph of N verticies can have?
Important Figures	 Fig. 1
VOCAB: (w/definition)	Clique - a subset of vertices such that every pair of vertices in the clique are adjacent
Cited references to	Z. Füredi, The number of maximal independent sets in connected graphs, preprint.

follow up on	J.W. Moon and L. Moser, On cliques in graphs, Israel J. Math. 3(1) (1965) 23-28.
Follow up Questions	Do graphs with cliques generally have less MIS, more MIS, or is there no correlation? What do the mentioned extremal graphs look like? What commonalities can be found in extremal graphs with different numbers of vertices? What are the actual applications of counting MIS, if any?

Article #6 Notes: The Number of Maximal Independent Sets in a Tree

Source Title	The Number of Maximal Independent Sets in a Tree
Source citation (APA Format)	Wilf, H. S. (1986). The number of maximal independent sets in a tree. *SIAM Journal on Algebraic Discrete Methods*, *7*(1), 125–130. https://doi.org/10.1137/0607015
Original URL	https://www2.math.upenn.edu/~wilf/website/Maximal%20independent%20sets%20in%20a%20tree.pdf
Source type	Journal Article
Keywords	Cliques, Trees, MIS
#Tags	#Counting #MIS
Summary of key points + notes (include methodology)	$f(n)=\begin{cases} 2^{n/2-1}+1 & \text{if } n \geqq 2 \text{ is even,} \\ 2^{(n-1)/2} & \text{if } n \text{ is odd,} \\ 1 & \text{if } n=0, \end{cases}$ Piecewise-defined function for the number of MIS in a tree with given number of vertices Optimal trees are batons in disguise! Methodology here is quite understandable. They create a diagram that can fit any tree and then make general statements about to that lead to properties and a proof Peacewise defined function for # of MIS i
Research Question/Problem/ Need	How many MIS can be in a tree, what's that optimal tree look like, and how can one prove it?
Important Figures	 n ODD n EVEN FIG. 1 Optimal trees (Star graphs, also batons)

	 FIG. 2 Generalized tree graphs! Super cool methodology
VOCAB: (w/definition)	Rooted (of a subtree) oriented with one node ‘down’ and the rest ‘up’. Can be used to generate subtrees
Cited references to follow up on	E. LAWLER, A note on the complexity of the chromatic number problem, Inform. Proc. Lett., 5 (1976), pp. 66-67.
Follow up Questions	Can the way they made a generalized tree graph be used for other types of graph to count MIS? Would forest graphs have more MIS than tree graphs (with the same number of verticies)? Generally, do all types of graph with N verticies have an optimal graph with a max number of MIS that scales with 2^N? Are there types of graph that scale worse than 2^N? Better?

Article #7 Notes: A note on the complexity of the chromatic number problem

Source Title	A note on the complexity of the chromatic number problem
Source citation (APA Format)	Lawler, E. L. (1976). A note on the complexity of the chromatic number problem. *Information Processing Letters*, *5*(3), 66–67. doi:10.1016/0020-0190(76)90065-X
Original URL	https://www.sciencedirect.com/science/article/pii/002001907690065X
Source type	Journal Article
Keywords	Graph, algorithm, chromatic number, complexity
#Tags	#Counting #Independent
Summary of key points + notes (include methodology)	Chromatic Number Problem – Counting the minimum number of independent sets needed to fully cover a given graph. A graph is said to be k-colorable if it has a chromatic number of k. Calculating the minimum k is NP-complete, and the worst-case running time for a graph is O(mn(1+cbrt(3))^n) where m is the number of edges and n is the number of vertices
Research Question/Problem/ Need	Can an upper bound for computation time for an algorithm to solve the chromatic number problem on graphs be created?
Important Figures	There weren't any figures in this paper. Instead, here are the most important equations: $O(mn(1+\sqrt[3]{3})^n)$, where m is the number of arc the graph and n is the number of nodes. (Note: $1+\sqrt[3]{3} \approx 2.445$.) $\chi(N') = 1 + \chi(N' - S)$. $\chi(N') = 1 + \min_{S \subseteq N'} \{\chi(N' - S)\}, \quad N' \neq \emptyset,$
VOCAB: (w/definition)	k-coloring – a system of MIS that cover a graph k-colorable – a graph with a k-coloring Chromatic Number – the least k such that a graph is k-colorable
Cited references	F. Gavril, Algorithms for Minimum Coloring, Maximum Clique, Minimum Covering by

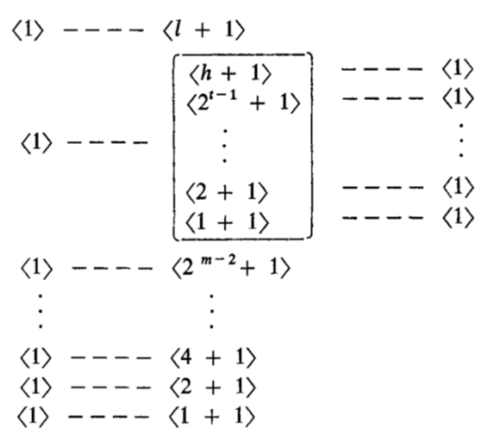
to follow up on	Cliques, and Maximum Independent Set of a Chordal Graph, SIAM J. Comput. 1(l972) 180-187. M.R. Carey and D.S. Johnson, The Complexity of Near-Optimal Graph Coloring, J. ACM 23 (1976) 43-49.
Follow up Questions	What's the largest chromatic number of a graph with a given number of sides? Are all planar graphs 4-colorable? (There's a 4-color theorem for maps which relates I believe) Are k-colorable graphs k-partite and vice versa?

Article #8 Notes: Tree-chromatic number

Source Title	Tree-chromatic number
Source citation (APA Format)	Seymour, P. (2016). Tree-chromatic number. *Journal of Combinatorial Theory, Series B*, *116*, 229–237. https://doi.org/10.1016/j.jctb.2015.08.002
Original URL	https://www.sciencedirect.com/science/article/pii/S0095895615001069
Source type	Journal Article
Keywords	Induced subgraph, Coloring, Tree-decomposition
#Tags	#Counting #Independent
Summary of key points + notes (include methodology)	A graph *G* has “tree-chromatic number” of *k* if the smallest chromatic number of any tree decomposition of G is k. All chromatic numbers >2 are possible (1 is impossible for any connected graph). This paper was just exploring interesting properties of this new concept, and didn’t really work towards a goal.
Research Question/Problem/ Need	How can chromatic numbers be applied to trees in a nontrivial way? (bc all trees normally have a chromatic number ≤2)
Important Figures	This paper didn’t have any figures. Here are some central equations instead: **1.1.** *For every graph G, there is a separation (A, B) of G such that* $\chi(A \cap B$ *and* $$\chi(A \setminus B), \chi(B \setminus A) \geq \chi(G) - \Upsilon(G).$$ $$s = \log(k) + \frac{1}{2}\log\log(k) + \frac{1}{2}\log(\pi/2) + o(1),$$ $$v \in Q_r^+ \cap Q_t \subseteq Q_s \subseteq X_s$$
VOCAB: (w/definition)	tree-decomposition – a process that converts a graph into a tree by grouping neighboring vertices into a new vertex tree-chromatic number – a ‘chromatic version’ of treewidth, the smallest chromatic number over all tree-decompositions

Cited references to follow up on	P. Erdős, Graph theory and probability, Canad. J. Math. 11 (1959) 34–38. T. Huynh, R. Kim, Tree-chromatic number is not equal to path-chromatic number, arXiv:1505.06234, May 2015. A. Kosowski, B. Li, N. Nisse, K. Suchan, k-chordal graphs: from cops and robber to compact routing via treewidth, Algorithmica 72 (2015) 758–777; also in: Automata, Languages, and Programming, Springer, 2012, pp. 610–622.
Follow up Questions	What's the maximum number of MIS that can be on a tree-decomposition of a graph with a given number of verticies? What happens when tree-decomposition is applied to tori graphs? Can tree-decomposition be recursively applied? What would happen? What properties might come from defining a new function as the number of decompositions it takes for a graph to become a single vertex?

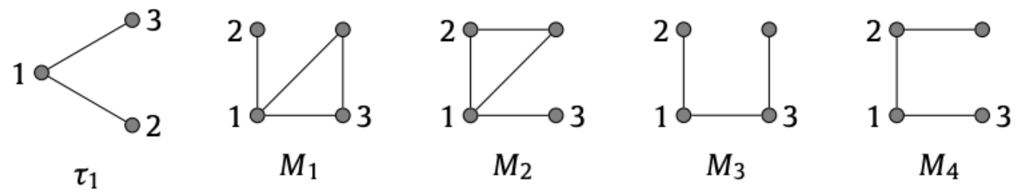
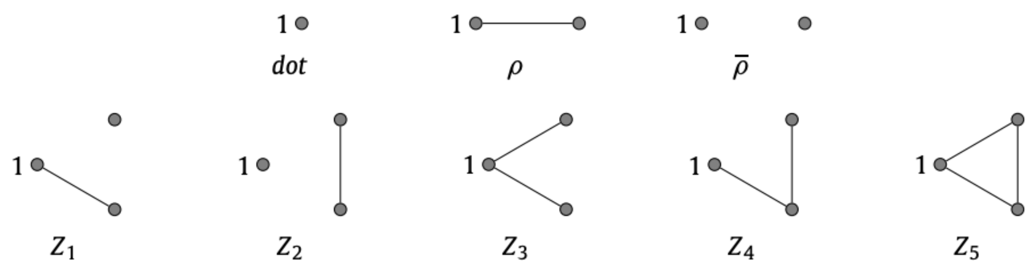
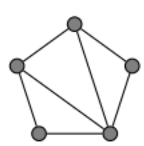
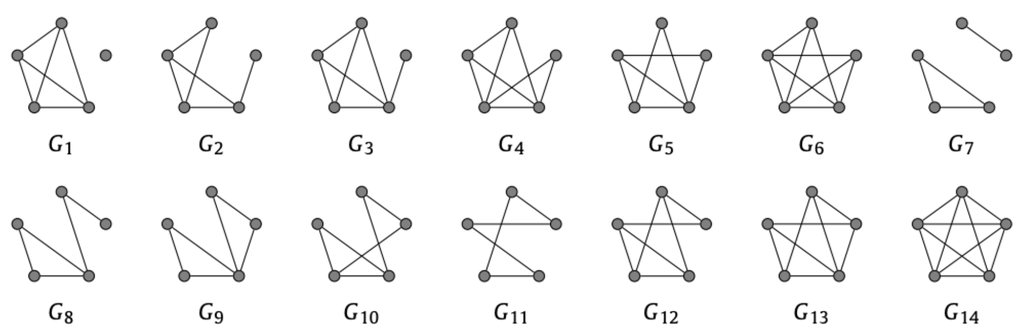
Article #9 Notes: On Cliques in Graphs

Source Title	On Cliques in Graphs
Source citation (APA Format)	Moon, J.W., & Moser, L. (1965). On cliques in graphs. *Israel Journal of Mathematics, 3*, 23-28.
Original URL	https://users.monash.edu.au/~davidwo/MoonMoser65.pdf
Source type	Journal Article
Keywords	Clique, Subgraph, Complete
#Tags	#Counting, #Cliques
Summary of key points + notes (include methodology)	Bounds were found for the number of different sizes of maximal cliques that could be made and it was shown that $$g(n) \sim n - [\log_2 \ n].$$ Maximum number of maximal cliques possible is given by this piecewise equation $$\textit{If } n \geqq 2, \textit{ then } f(n) = \begin{cases} 3^{n/3}, & \text{if } n \equiv 0 \pmod 3; \\ 4.3^{[n/3]-1}, & \text{if } n \equiv 1 \pmod 3; \\ 2.3^{[n/3]}, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$ Methodology made use of proof by contradiction, starting with the opposite of the hypothesis and deriving a contradiction.
Research Question/Problem/ Need	What’s the max number of maximal cliques that can be in a graph with N vertices?
Important Figures	

VOCAB: (w/definition)	Clique - a subgraph such that every pair of vertices in the clique are adjacent.
Cited references to follow up on	P. Turin, On the theory of graphs, Colloq. Math. 3 0954), 19-30.
Follow up Questions	What graphs optimize the largest number of order-n cliques for any given n? Given a certain number of edges that must be placed on a given number of vertices, what's the largest number of cliques that can be created? Given a certain number of edges that must be placed on a given number of vertices, what's the smallest possible largest clique that can be created?

Article #10 Notes: A problem of Erdős on the minimum number of k-cliques

Source Title	A problem of Erdős on the minimum number of k-cliques
Source citation (APA Format)	Das, S., Huang, H., Ma, J., Naves, H., & Sudakov, B. (2013). A problem of erdős on the minimum number of K -Cliques. *Journal of Combinatorial Theory, Series B*, *103*(3), 344–373. https://doi.org/10.1016/j.jctb.2013.02.003
Original URL	https://www.sciencedirect.com/science/article/pii/S0095895613000142
Source type	Journal Article
Keywords	Clique density, Flag algebras, independence number, flag algebra
#Tags	#Counting #Cliques
Summary of key points + notes (include methodology)	This paper aimed to find the minimum number of k-cliques in a graph on n vertices with independence number less than l for (k,l) = (3,4) and (4,3) It did this by reducing complicated graph structures into algebraic ones through a flag algebra, essentially decomposing a graph into smaller, simpler, graphs. Here's some examples of said decomposition:

	 It ended up finding that the optimal graph is the graph obtained by 1. Starting with a 5-cycle 2. replace every vertex with a finite collection of copies so that the copies of two vertices are adjacent if and only if the originals are
Research Question/Problem/ Need	What's the minimum number of k-cliques in a graph on n vertices with independence number less than l for some given (k,n,l)?
Important Figures	 **Fig. 1.** Some examples of flags of type *dot*.  **Fig. 2.** Graph W.  **Fig. 3.** Graphs of size 5 with independence number at most 2.

VOCAB: (w/definition)	Flag algebra - a method that turns problems about large structures into algebraic ones by analyzing small substructures (flags) and their interactions. Clique density - the proportion of cliques of a given size within a graph, compared to the total number of possible cliques of that size
Cited references to follow up on	N. Alon, J. Spencer, The Probabilistic Method, John Wiley Inc., New York, 2008. P. Erdos, On the number of complete subgraphs contained in certain graphs, Publ. Math. Inst. Hungar. Acad. Sci. 7 (1962) ˝ 459–464.
Follow up Questions	Can clique density be calculated for random graphs and analyzed as a potential project idea? Can flag algebras be used to improve efficiency of counting algorithms by removing the need for counting isomorphic graphs? Do graphs with higher clique density have lower independence numbers on average?

Article #11 Notes: K-th Order Maximal Independent Sets in Path and Cycle Graphs

Source Title	k-th order maximal independent sets in path and cycle graphs
Source citation (APA Format)	Yanco, R., & Bagchi, A. (1994). K-th order maximal independent sets in path and cycle graphs. Retrieved from https://oeis.org/A007380/a007380_1.pdf
Original URL	https://oeis.org/A007380/a007380_1.pdf
Source type	Mathematical Paper
Keywords	Order-K MIS, MIS, Lukas sequence, Perrin Sequence, Path Graph, Cycle Graph
#Tags	#Counting #MIS
Summary of key points + notes (include methodology)	Define order-k MIS as an MIS with no MIS of larger cardinality with hamming distance ≤ k from it Only odd k matter, even are duplicates # of Order-k MIS on Path & Cycle graphs form recursive sequences PRIME PROPERTY – p | a_p for all prime p
Research Question/Problem / Need	How many order-k MIS exist on path and cycle graphs?
Important Figures	

Path Length	Number of Colored Vertices 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	$\lvert XP_{3,n}\rvert$
1	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1
2	2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	2
3	-	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1
4	-	3	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3
5	-	1	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	2
6	-	-	4	-	-	-	-	-	-	-	-	-	-	-	-	-	-	4
7	-	-	3	1	-	-	-	-	-	-	-	-	-	-	-	-	-	4
8	-	-	-	5	-	-	-	-	-	-	-	-	-	-	-	-	-	5
9	-	-	-	6	1	-	-	-	-	-	-	-	-	-	-	-	-	7
10	-	-	-	1	6	-	-	-	-	-	-	-	-	-	-	-	-	7
11	-	-	-	-	10	1	-	-	-	-	-	-	-	-	-	-	-	11
12	-	-	-	-	4	7	-	-	-	-	-	-	-	-	-	-	-	11
13	-	-	-	-	-	15	1	-	-	-	-	-	-	-	-	-	-	16
14	-	-	-	-	-	10	8	-	-	-	-	-	-	-	-	-	-	18
15	-	-	-	-	-	1	21	1	-	-	-	-	-	-	-	-	-	23
16	-	-	-	-	-	-	20	9	-	-	-	-	-	-	-	-	-	29
17	-	-	-	-	-	-	5	28	1	-	-	-	-	-	-	-	-	34
18	-	-	-	-	-	-	-	35	10	-	-	-	-	-	-	-	-	45
19	-	-	-	-	-	-	-	15	36	1	-	-	-	-	-	-	-	52
20	-	-	-	-	-	-	-	1	56	11	-	-	-	-	-	-	-	68
21	-	-	-	-	-	-	-	-	35	45	1	-	-	-	-	-	-	81
22	-	-	-	-	-	-	-	-	6	84	12	-	-	-	-	-	-	102
23	-	-	-	-	-	-	-	-	-	70	55	1	-	-	-	-	-	126
24	-	-	-	-	-	-	-	-	-	21	120	13	-	-	-	-	-	154
25	-	-	-	-	-	-	-	-	-	1	126	66	1	-	-	-	-	194
26	-	-	-	-	-	-	-	-	-	-	56	165	14	-	-	-	-	235
27	-	-	-	-	-	-	-	-	-	-	7	210	78	1	-	-	-	296
28	-	-	-	-	-	-	-	-	-	-	-	126	220	15	-	-	-	361
29	-	-	-	-	-	-	-	-	-	-	-	28	330	91	1	-	-	450
30	-	-	-	-	-	-	-	-	-	-	-	1	252	286	16	-	-	555
31	-	-	-	-	-	-	-	-	-	-	-	-	84	495	105	1	-	685
32	-	-	-	-	-	-	-	-	-	-	-	-	8	462	364	17	-	851
33	-	-	-	-	-	-	-	-	-	-	-	-	-	210	715	120	1	1046
34	-	-	-	-	-	-	-	-	-	-	-	-	-	36	792	455	18	1301
35	-	-	-	-	-	-	-	-	-	-	-	-	-	1	462	1001	136	1601
36	-	-	-	-	-	-	-	-	-	-	-	-	-	-	120	1287	560	1986

Cycle Length	Number of Colored Vertices 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	$\lvert XC_{3,n} \rvert$
1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	0
2	2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	2
3	3	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3
4	-	2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	2
5	-	5	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	5
6	-	-	2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	2
7	-	-	7	-	-	-	-	-	-	-	-	-	-	-	-	-	-	7
8	-	-	-	2	-	-	-	-	-	-	-	-	-	-	-	-	-	2
9	-	-	-	9	-	-	-	-	-	-	-	-	-	-	-	-	-	9
10	-	-	-	5	2	-	-	-	-	-	-	-	-	-	-	-	-	7
11	-	-	-	-	11	-	-	-	-	-	-	-	-	-	-	-	-	11
12	-	-	-	-	12	2	-	-	-	-	-	-	-	-	-	-	-	14
13	-	-	-	-	-	13	-	-	-	-	-	-	-	-	-	-	-	13
14	-	-	-	-	-	21	2	-	-	-	-	-	-	-	-	-	-	23
15	-	-	-	-	-	5	15	-	-	-	-	-	-	-	-	-	-	20
16	-	-	-	-	-	-	32	2	-	-	-	-	-	-	-	-	-	34
17	-	-	-	-	-	-	17	17	-	-	-	-	-	-	-	-	-	34
18	-	-	-	-	-	-	-	45	2	-	-	-	-	-	-	-	-	47
19	-	-	-	-	-	-	-	38	19	-	-	-	-	-	-	-	-	57
20	-	-	-	-	-	-	-	5	60	2	-	-	-	-	-	-	-	67
21	-	-	-	-	-	-	-	-	70	21	-	-	-	-	-	-	-	91
22	-	-	-	-	-	-	-	-	22	77	2	-	-	-	-	-	-	101
23	-	-	-	-	-	-	-	-	-	115	23	-	-	-	-	-	-	138
24	-	-	-	-	-	-	-	-	-	60	96	2	-	-	-	-	-	158
25	-	-	-	-	-	-	-	-	-	5	175	25	-	-	-	-	-	205
26	-	-	-	-	-	-	-	-	-	-	130	117	2	-	-	-	-	247
27	-	-	-	-	-	-	-	-	-	-	27	252	27	-	-	-	-	306
28	-	-	-	-	-	-	-	-	-	-	-	245	140	2	-	-	-	387
29	-	-	-	-	-	-	-	-	-	-	-	87	348	29	-	-	-	464
30	-	-	-	-	-	-	-	-	-	-	-	5	420	165	2	-	-	592
31	-	-	-	-	-	-	-	-	-	-	-	-	217	465	31	-	-	713
32	-	-	-	-	-	-	-	-	-	-	-	-	32	672	192	2	-	898
33	-	-	-	-	-	-	-	-	-	-	-	-	-	462	605	33	-	1100
34	-	-	-	-	-	-	-	-	-	-	-	-	-	119	1020	221	2	1362
35	-	-	-	-	-	-	-	-	-	-	-	-	-	5	882	770	35	1692
36	-	-	-	-	-	-	-	-	-	-	-	-	-	-	336	1485	252	2075

VOCAB: (w/definition)

k-neighborhood – all graphs with hamming distance ≤ k from a given graph

Order-K MIS – a MIS with no larger-cardinality MIS in its k-neighborhood

Cited references to follow up on

A. Bagchi and A.C. Williams, k-order local optima and recognition problems for certain classes of pseudo-Boolean functions. Technical Report, RUTCOR, Rutgers Unitversity (1990).

Follow up Questions

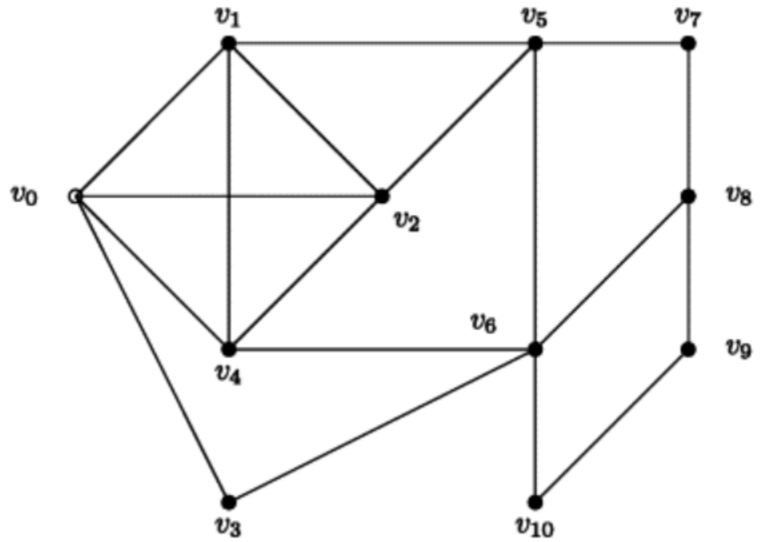
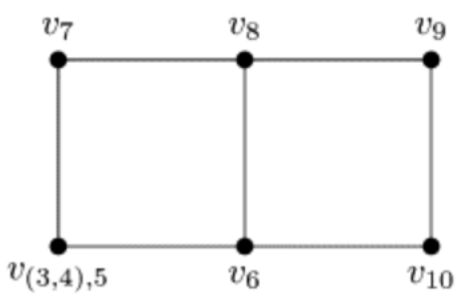
How many exist on other graph types?

How many maximal, but non-order-k maximal sets are there for different k's on different graphs?

What do these sets look like on a graph? Can one be identified by looking at certain structures in the set?

Article #12 Notes: Pseudo-Boolean Optimization

Source Title	Pseudo-Boolean Optimization
Source citation (APA Format)	Boros, E., & Hammer, P. L. (2002). Pseudo-Boolean optimization. *Discrete Applied Mathematics, 123*(1-3), 155-225. https://doi.org/10.1016/S0166-218X(01)00341-9
Original URL	https://www.sciencedirect.com/science/article/pii/S0166218X01003419#aep-article-footnote-id9
Source type	Journal Article
Keywords	Pseudo-Boolean functions, optimization, set functions, multi-linear polynomials, algorithmic graph theory
#Tags	#Optimization # Algorithm
Summary of key points + notes (include methodology)	- The paper surveys problems and methodologies in pseudo-Boolean optimization, focusing on functions represented by closed algebraic expressions. - Quadratic pseudo-Boolean optimization is highlighted as a key area since all pseudo-Boolean problems can reduce to this case. - Applications range from VLSI design, statistical mechanics, and computer science to economics, game theory, and physics. - Specific attention is paid to mathematical and algorithmic techniques for optimization, including convex/nonconvex analysis and nonlinear programming.
Research Question/Problem/ Need	How can pseudo-Boolean functions be efficiently optimized, particularly for special cases like quadratic, submodular, and hyperbolic functions?

Important Figures	 
VOCAB: (w/definition)	Pseudo-Boolean Function: A set function represented by an algebraic expression mapping subsets of a finite set (which can model MIS) to real values (which can model number of verticies in a set) Multi-Linear Polynomial: Polynomial representation of a pseudo-Boolean function where each term is a product of variables raised to the first power
Cited references to follow up on	These numbers are linked to papers at the bottom of the article [72], [94], [95]: Studies on the optimization of pseudo-Boolean functions. [17], [50]: APPLICATIONS in VLSI design and statistical mechanics. [115], [126], [167]: Pseudo-Boolean functions in physics, more applications?
Follow up Questions	What are the computational limitations of deriving closed-form algebraic representations for pseudo-Boolean functions? How do approximation algorithms for pseudo-Boolean optimization work? How effective are they?

	Can pseudo-Boolean optimization models be applied to efficiently count MIS? Order-k MIS?

Article #13 Notes: Logic Programming with Pseudo-Boolean Constraints

Source Title	Logic Programming with Pseudo-Boolean Constraints
Source citation (APA Format)	Hansen, P. (1979). Methods of nonlinear 0-1 programming. In *Annals of Discrete Mathematics* (Vol. 5, pp. 53-70). Elsevier.
Original URL	https://pure.mpg.de/rest/items/item_1834962_3/component/file_1857491/content
Source type	Journal Article
Keywords	Pseduo-boolean function, boolean, integer, logic programming
#Tags	#Counting #Coding #Algorithm
Summary of key points + notes (include methodology)	The author introduces CLP(PB), a constraint logic programming language specifically for pseudo-Boolean functions (booleans in, integer out). It's a specialization of the CLP(X) framework, combining arithmetic and Boolean algebra. LP(PB) retains the general semantic properties of the CLP(X) scheme. The author shows Pseudo-Boolean unification techniques, which create equivalences between different logical constraints (applications to simplifying specific cases of MIS counting?) He introduces a basic algorithm for optimization, as well as a specialized version for specific cases Constraints added could include independent & maximality
Research Question/Problem / Need	How can pseudo-Boolean constraints be added into logical programming frameworks and how can algorithms be developed to solve and optimize these constraints efficiently (in P time hopefully)?

Important Figures

$$
\begin{aligned}
B = \{X \oplus (Y \oplus Z) &\doteq (X \oplus Y) \oplus Z, \\
X \oplus 0 &\doteq X, \\
X \oplus X &\doteq 0, \\
X \oplus Y &\doteq Y \oplus X, \\
X \wedge (Y \wedge Z) &\doteq (X \wedge Y) \wedge Z, \\
X \wedge 1 &\doteq X, \\
X \wedge Y &\doteq Y \wedge X, \\
X \wedge (Y \oplus Z) &\doteq (X \wedge Y) \oplus (X \wedge Z), \\
X \wedge X &\doteq X, \\
\overline{X} &\doteq X \oplus 1, \\
X \vee Y &\doteq (X \oplus Y) \oplus X \wedge Y\}
\end{aligned}
$$

```
assemble(T1, T2, T3, F, S, W, P) :-
                T1 + T2 + T3 = 1,
                W + P = 1,
                F + S = 1,
                P ≤ T2 * S,       % P = 1 implies T2 = S = 1
                T1 ≤ F,           % T1 = 1 implies F = 1
                T2 ≤ S,           % T2 = 1 implies S = 1
                T3 ≤ S.           % T3 = 1 implies S = 1

maximal-profit(T1, T2, T3, F, S, W, M) :-
        max( 110W + 105M - ( 28T1 + 30T2 + 31T3 + 25F + 23S
                           + 9W + 6P + 27T1 + 28T2 + 25T3 + 10),
        assemble(T1, T2, T3, F, S, W, M) ).
```

$$
\begin{aligned}
\sigma(f) &= \sigma(X_{n+1}) * \sigma(g) + \sigma(h) \\
&= (\rho(g^{=}) * Y + \rho(g^{>})) * \rho(g) + \rho(h) \\
&= \rho(g^{=}) * \rho(g) * Y + \rho(g^{>}) * \rho(g) + \rho(h) \\
&= \rho(g^{=} * g) * Y + \rho(g^{>} * g) + \rho(h) \\
&= \rho(0) * Y + \rho(g^{+}) + \rho(h) \\
&= 0 + z = z.
\end{aligned}
$$

Here we have used the identities $g^{=} * g = 0$ and $g^{>} * g = g^{+}$. This shows, that σ is a maximizer of f.

VOCAB:

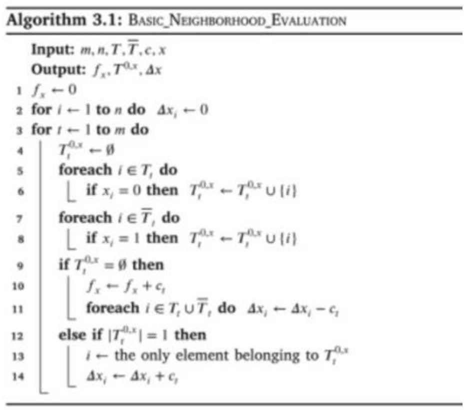
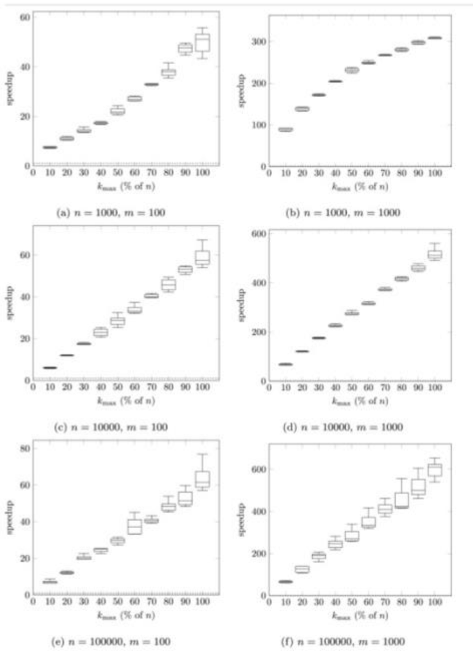
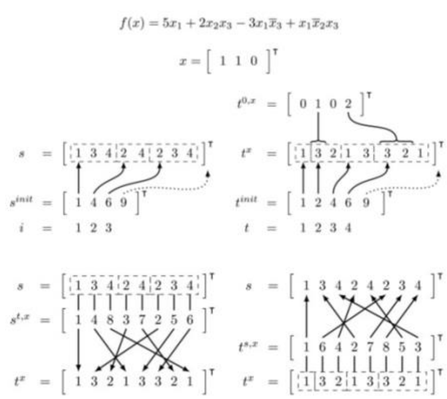
A **pseudo-Boolean function** is an integer-valued function defined on Boolean inputs, like

(w/definition)	determining the order of a maximal independent set. It's a framework in which I could study functions on sets in graphs
Cited references to follow up on	P.L. Hammer, I. Rosenberg, and S. Rudeanu. On the determination of minima of pseudo-boolean functions. *Stud. Cerc. Mat.*, 14:359–364, 1963. (in Romanian). P.L. Hammer and S. Rudeanu. *Boolean Methods in Operations Research and Related Areas.* Springer-Verlag, 1968.
Follow up Questions	How can pseudo-Boolean constraints encode Independence and/or maximality? Are there special cases (specific path lengths or cycle sizes) where optimizations simplify significantly? How do the proposed algorithms scale with the size of the input space (NP? P?)?

Article #14 Notes: Fast 1-flip neighborhood evaluations for large-scale pseudo-Boolean optimization using posiform representation

Source Title	Fast 1-flip neighborhood evaluations for large-scale pseudo-Boolean optimization using posiform representation
Source citation (APA Format)	Liang, R. N., Anacleto, E. A. J., & Meneses, C. N. (2023). Fast 1-flip neighborhood evaluations for large-scale pseudo-boolean optimization using posiform representation. *Computers & Operations Research*, *159*, 106324. https://doi.org/10.1016/j.cor.2023.106324
Original URL	https://www.sciencedirect.com/science/article/pii/S0305054823001880
Source type	Journal Article
Keywords	Flip, Pseduo-boolean function, boolean, integer,
#Tags	#Counting #Coding #Algorithm
Summary of key points + notes (include methodology)	1-flip neighborhoods are the same as order-3 maximal, so this paper talks about similar restrictions as I'm looking into, and explains that in the framework of pseudo-Boolean functions. The algorithms shown are pretty efficient for larger graphs
Research Question/Problem/	How can fast 1-flip neighborhood evaluation techniques and posiform representations be adapted to efficiently enumerate order- k k maximal independent sets in Cartesian

Need	products of paths and cycles?

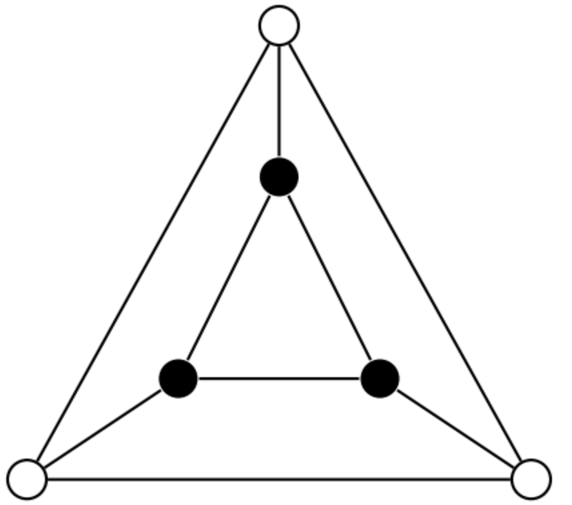
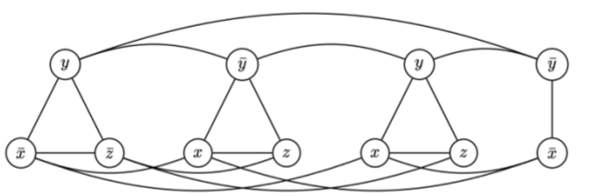
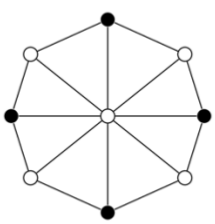
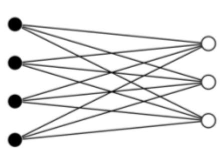
Important Figures	
	Algorithm 3.1: BASIC_NEIGHBORHOOD_EVALUATION
	Input: $m, n, T, \overline{T}, c, x$
	Output: $f_x, T^{0,x}, \Delta x$
	$f_x \leftarrow 0$
	for $i \leftarrow 1$ **to** n **do** $\Delta x_i \leftarrow 0$
	for $t \leftarrow 1$ **to** m **do**
	$T_t^{0,x} \leftarrow \emptyset$
	foreach $i \in T_t$ **do**
	if $x_i = 0$ **then** $T_t^{0,x} \leftarrow T_t^{0,x} \cup \{i\}$
	foreach $i \in \overline{T}_t$ **do**
	if $x_i = 1$ **then** $T_t^{0,x} \leftarrow T_t^{0,x} \cup \{i\}$
	if $T_t^{0,x} = \emptyset$ **then**
	$f_x \leftarrow f_x + c_t$
	foreach $i \in T_t \cup \overline{T}_t$ **do** $\Delta x_i \leftarrow \Delta x_i - c_t$
	else if $\|T_t^{0,x}\| = 1$ **then**
	$i \leftarrow$ the only element belonging to $T_t^{0,x}$
	$\Delta x_i \leftarrow \Delta x_i + c_t$
	
	
	

VOCAB: (w/definition)	Pseudo-Boolean Optimization (PBO) - A type of optimization problem where the objective function is a polynomial of binary variables (values are 0 or 1) (NP Hard) Quadratic Unconstrained Binary Optimization (QUBO) - A specific type of PBO where the objective function is a quadratic polynomial. It has a lot of applications in combinatorial optimization and quantum computing. 1-Flip Move - A change in which a single binary variable flips its value (from 0 to 1 or 1 to 0). Used in optimization methods to explore neighboring solutions. Neighborhood Evaluation - The process of calculating the effect of a 1-flip move on the objective function value for a given solution
Cited references to	Glover, F., & Hao, J. K. (2010). Efficient evaluations for solving large 0-1 unconstrained quadratic optimisation problems. *International Journal of Metaheuristics*, *1*(1), 3-10.

follow up on	Fraenkel, A. S., & Hammer, P. L. (1984). Pseudo-Boolean functions and their graphs. *Annals of Discrete Mathematics*, *20*, 137-146. Tinós, R., Whitley, D., & Chicano, F. (2015, January). Partition crossover for pseudo-boolean optimization. In *Proceedings of the 2015 ACM Conference on Foundations of Genetic Algorithms XIII* (pp. 137-149).
Follow up Questions	Can the 1-flip evaluation techniques be adapted to efficiently find or count order-k maximal independent sets? How might the posiform representation be used to model or analyze independence constraints in graphs? Could an Iterated Tabu Search or similar algorithm be used to explore the space of independent sets efficiently?

Article #15 Notes: Independent Set, Clique, and Vertex Cover

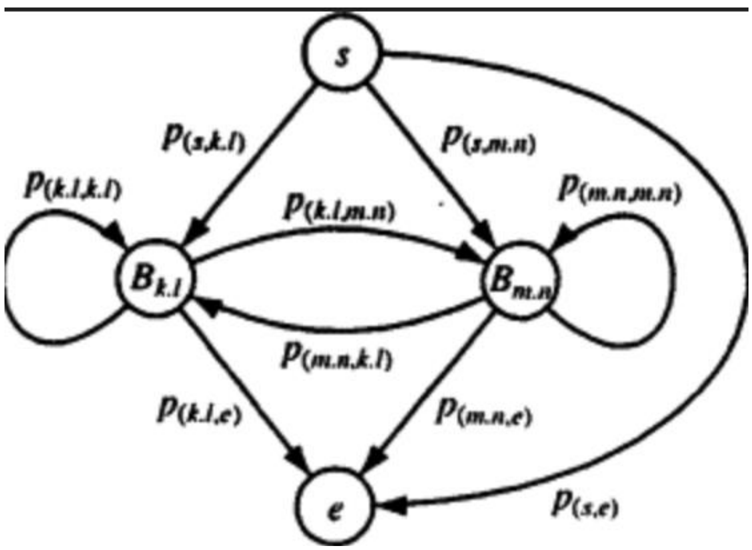
Source Title	Independent Set, Clique, and Vertex Cover
Source citation (APA Format)	Ladha, A. (2024, March 26). *CS 3510 Algorithms 3/26/2024 Lecture 15: Independent Set, Clique, and Vertex Cover* [Lecture notes]. Scribed by Richard Zhang.
Original URL	https://faculty.cc.gatech.edu/~ladha/S24/3510/L15.pdf
Source type	Lecture Notes
Keywords	Independent Set, Vertex Cover, Clique, Set, Graph
#Tags	#Counting #Coding #Algorithm #Clique #MIS
Summary of key points + notes (include methodology)	The lecture focuses on the computational complexity of three closely related graph problems: Independent Set, Clique, and Vertex Cover, all of which are proven to be NP-complete. A Maximal Independent Set is the same as a set which is both a vertex cover and an independent set. Just because finding two sets of sets is NP-complete though, doesn't necessitate that finding their intersection is NP-complete. The lecture further discusses graph compliments and proves NP-completeness for each of the central problems.

Research Question/Problem/ Need	Why are the Independent Set, Clique, and Vertex Cover problems so hard to compute, how can computational complexity be quantified, and how can we prove equivalencies in computational complexity?
Important Figures	 Figure 5: Clique of a graph  Figure 3: Constructed graph of formula $(\bar{x} \vee y \vee \bar{z}) \wedge (x \vee \bar{y} \vee z) \wedge (x \vee y \vee z) \wedge (\bar{x} \vee \bar{y})$  Figure 1: Independent set of Wheel graph  Figure 2: Independent set of Bipartite Graph
VOCAB: (w/definition)	Vertex Cover – A set of vertices S such that every vertex is either in S or connected to a vertex in S

	Clique - A complete subgraph of a graph, where every vertex is connected to every other vertex in the subgraph NP-complete: A classification of problems in computational theory that are both in NP and as hard as any problem in NP. Any one problem being P hard would make all problems P hard.
Cited references to follow up on	This source unfortunately doesn't have any cited references to follow up on.
Follow up Questions	Can properties of the Cartesian product of path and cycle graphs be leveraged to reduce the effective search space and make the complexity better? Can vertex cover or clique formulations provide alternative approaches to counting MIS? How are efficient algorithms designed for specific graph products, even if the problem is NP-complete for general graphs?

Article #16 Notes: Pseudo-Boolean Functions and Stability of Graphs

Source Title	Pseudo-Boolean Functions and Stability of Graphs
Source citation (APA Format)	Ebenegger, Ch., Hammer, P. L., & de Werra, D. (1984). Pseudo-boolean functions and stability of graphs. *North-Holland Mathematics Studies*, *95*, 83–97. https://doi.org/10.1016/s0304-0208(08)72955-4
Original URL	https://www.sciencedirect.com/science/article/abs/pii/S0304020808729554
Source type	Journal Article
Keywords	Conflict graph – a graph where vertices represent inputs and edges represent conflicts (inabilities to have two inputs simultaneously true)
#Tags	#Coding #Algorithm
Summary of key points + notes (include methodology)	Pseudo-Boolean Functions map binary variables (0 or 1) to real numbers, optimizing stable sets (independent sets) in graphs. Conflict graphs represent conflicts between variables (vertices) and are used to model

	and optimize stable sets. Finding the maximum weight stable set in a graph can be reduced to maximizing a pseudo-Boolean function, which also determines the stability number (size of the largest independent set). A procedure transforms the graph into one with a smaller stability number, simplifying the problem. Preliminary computational experiments suggest that pseudo-Boolean functions improve algorithm efficiency for calculating the stability number.
Research Question/Problem/ Need	How can the application of pseudo-Boolean functions improve the efficiency of algorithms for calculating the stability number of graphs?
Important Figures	
VOCAB: (w/definition)	- Conflict Graph: A graph where vertices represent variables, and edges represent conflicts between variables that cannot be true simultaneously. - Stable Set: A set of vertices in a graph such that no two vertices in the set are adjacent (also called an independent set). - Stability Number: The size of the largest stable set in a graph. - Graph Transformation: A procedure that changes a graph into another form, often to simplify or optimize certain problems.
Cited references to follow up on	Chvátal, V. (1975). On certain polytopes associated with graphs. *Journal of Combinatorial Theory Series B*, *18*(2), 138–154. https://doi.org/10.1016/0095-8956(75)90041-6
Follow up Questions	What impact does reducing a graph's stability number using the proposed transformation have on the number of MIS? Order-k MIS? Can the pseudo-Boolean function approach be adapted for counting weighted maximal

	independent sets? How efficient is the algorithm described in the article for large graphs or graphs of specific types (like product graphs)?

Article #17 Notes: A Simple Parallel Algorithm for the Maximal Independent Set Problem

Source Title	A Simple Parallel Algorithm for the Maximal Independent Set Problem
Source citation (APA Format)	Luby, M. (1985). A simple parallel algorithm for the maximal independent set problem. *Proceedings of the Seventeenth Annual ACM Symposium on Theory of Computing - STOC '85*, 1–10. https://doi.org/10.1145/22145.22146
Original URL	https://courses.csail.mit.edu/6.852/08/papers/Luby.pdf
Source type	Journal Article
Keywords	parallel computations, NC, maximal independent set, randomizing algorithms, pairwise independences
#Tags	#Coding #Algorithm
Summary of key points + notes (include methodology)	The paper introduces simple & fast parallel algorithms for solving the MIS problem Design strategies of the code: - Localized Parallel Execution - Copies of a simple algorithm are assigned to small, local parts of the input, which are executed simultaneously to produce the correct output efficiently - Removing Randomization - A general technique is used to convert a randomized Monte Carlo algorithm into a deterministic one without changing the parallel runtime Built off the Monte Carlo Algorithm, which is a probabilistic method designed to quickly solve the MIS problem in parallel
Research Question/Problem / Need	How can a probabilistic algorithm for counting MIS be made into a deterministic one to efficiently count MIS?

Important Figures

Algorithm	P-RAM Type	Processors	Time	Random bits
A	CRCW	$O(m)$	$EO(\log n)$	$EO(n(\log n)^2$
B	EREW	$O(m)$	$EO((\log n)^2)$	$EO(n \log n)$
C	EREW	$O(m)$	$EO((\log n)^2)$	$EO((\log n)^2)$
D	EREW	$O(n^2 \cdot m)$	$O((\log n)^2)$	none

```
begin
  I ← ∅
  compute n = |V|
  compute a prime q such that n ≦ q ≦ 2n
  G' = (V', E') ← G = (V, E)
  while G' ≠ ∅ do
  begin
    In parallel, ∀i ∈ V', compute d(i)
    In parallel, ∀i ∈ V'
      if d(i) = 0 then add i to I and delete i from V'.
    find i ∈ V' such that d(i) is maximum
    if d(i) ≧ n/16 then add i to I and let G' be the
      graph induced on the vertices V' − ({i} ∪ N({i}))
    else (∀i ∈ V', d(i) < n/16)
    begin
(choice)  randomly choose x and y such that 0 ≦ x, y ≦ q − 1
          X ← ∅
          In parallel, ∀i ∈ V',
          begin
            compute n(i) = ⌊q/2d(i)⌋
            compute l(i) = (x + y · i) mod q
            if l(i) ≦ n(i) then add i to X
          end
          I' ← X
          In parallel, ∀i ∈ X, j ∈ X,
            if (i, j) ∈ E' then
              if d(i) ≦ d(j) then I' ← I' − {i}
              else I' ← I' − {j}
```

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```
          I ← I ∪ I'
          Y ← I' ∪ N(I')
          G' = (V', E') is the induced subgraph on V' − Y.
    end
  end
end
```

VOCAB: (w/definition)

Monte Carlo Algorithm - A type of algorithm that uses randomness to solve problems

Deterministic Algorithm - An algorithm that always produces the same output for a given input without relying on randomization

Parallel Computations: A method where multiple processes or calculations are executed simultaneously to speed up problem-solving

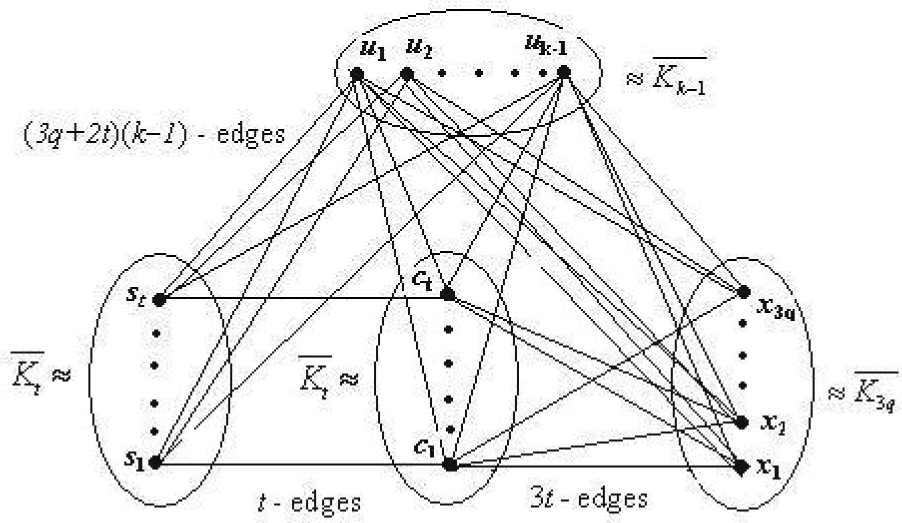
Cited references to follow up on

R.L. BROOKS, On colouring the nodes of a network, Proc. Cambridge Philos. Soc., 37 (1941), pp. 194-197.

	L. G. VALIANT, Parallel computation, Proc. 7th IBM Symposium on Mathematical Foundations of Computer Science, 1982. H. KARLOFF, D. SHMOYS AND D. SOROKER, Efficient parallel algorithms for graph coloring and partitioning problems, preprint.
Follow up Questions	How does the localized execution and parallel computation strategy help improve efficiency? What are the advantages of converting a Monte Carlo algorithm into a deterministic algorithm? Why is parallel computation well suited for solving graph problems like counting MIS? Are order-k MIS able to be calculated with a similar probabilistic approach?

Article #18 Notes: Perfect k-domination in Graphs

Source Title	Perfect k-domination in Graphs
Source citation (APA Format)	Chaluvaraju, B., Chellali, M., & Vidya, K. A. (2010). Perfect k-domination in graphs. *Australasian Journal of Combinatorics*, *48*, 175–184.
Original URL	https://ajc.maths.uq.edu.au/pdf/48/ajc_v48_p175.pdf
Source type	Journal Article
Keywords	Perfect k-domination NP-complete Perfect k-dominating set Domination number
#Tags	#Existance #Proof
Summary of key points + notes (include methodology)	The paper extends the concept of perfect domination in graphs to something called perfect k-domination. A perfect k-dominating set, D, is a subset of vertices where every vertex not in D is connected to exactly k vertices in D. The paper shows that finding a perfect k-dominating set is NP-complete, meaning it's exponentially computationally hard to solve for general graphs. The authors use proofs to establish bounds and to show the complexity of the problem.

	They prove NP-completeness by reducing the problem to other known NP-complete problems, which is a common way to show hardness.
Research Question/Problem/ Need	How can we efficiently determine the perfect k-domination number for general graphs, given that the problem is NP-complete?
Important Figures	 **Figure 1: NP-completeness constructions of k-perfect domination**
VOCAB: (w/definition)	Perfect k-dominating set - A subset of vertices in a graph where each vertex not in the subset is adjacent to exactly k vertices within the subset. Perfect k-domination number - The minimum size of a perfect k-dominating set in a graph G.
Cited references to follow up on	D.W. Bange, A. E. Barkauskas and P. J. Slater, Efficient dominating sets in graphs. Applications of discrete mathematics (Clemson, SC, 1986), 189–199, SIAM, Philadelphia, PA, 1988. J. Kratochvil and M. Krivanek, On the Computational Complexity of Codes in Graphs, in: Proc. MFCS (1988), LN in Comp. Sci. 324, Springer-Verlag, 396–404.
Follow up Questions	What are some real-world applications of perfect k-domination in graphs? Are there specific types of graphs (trees, planar graphs) where structure can be leveraged to make the problem is easier to solve? How do the bounds for perfect domination number compare with those for standard domination number in graphs?

Article #19 Notes: The Number of Maximal Independent Sets in the Hamming Cube

Source Title	The Number of Maximal Independent Sets in the Hamming Cube
Source citation (APA Format)	Kahn, J., & Park, J. (2019). The number of maximal independent sets in the Hamming cube. *arXiv preprint arXiv:1909.04283*.
Original URL	https://arxiv.org/abs/1909.04283
Source type	Journal Article
Keywords	- Hamming Cube - Maximal Independent Sets - Asymptotics - Isoperimetric Results - Induced Matchings
#Tags	#Counting #Bounds #Proof #MIS # Algorithm
Summary of key points + notes (include methodology)	The paper focuses on the n-dimensional Hamming cube, denoted Q_n , and its maximal independent sets. The authors prove an approximate value for the number of MIS in Q_n. This result confirms a conjecture by Ilinca and Kahn, tied to a question from Duffus, Frankl, and Rödl. The lower bound comes from analyzing the connection between maximal independent sets and induced matchings. The upper bound proof uses stability results and isoperimetric properties of Q_n .
Research Question/Problem/ Need	What is the asymptotic number of maximal independent sets in the n-dimensional Hamming cube Q_n?
Important Figures	$\lvert \mathcal{I}_1^* \cap \mathcal{I}_2^* \rvert \leq 3^{N/8}.$

	$$O(g\log n/n) = O(t\log n/\sqrt{n}).$$
VOCAB: (w/definition)	Hamming Cube - A graph representing binary strings of length n, where vertices differ by exactly one bit if and only if the verticies are connected by an edge Induced Matching - A subset of edges in a graph where no two edges share a vertex, and the vertices of these edges induce no other edges Isoperimetric Behavior - Describes how the surface area of a shape compares to its volume for different dimensions
Cited references to follow up on	P. Erdos, D.J. Kleitman and B.L. Rothschild, Asymptotic enumeration of ˝ Kn-free graphs, pp. 19-27 in Colloquio Internazionale sulle Teorie Combinatorie (Rome, 1973) Tomo II, Atti dei Convegni Lincei, No. 17, Accad. Naz. Lincei, Rome, 1976. A.A. Sapozhenko, The number of independent sets in graphs, pp. 116-118 in Moscow Univ. Math. Bull. 62, 2007. A.A. Sapozhenko, The number of antichains in ranked partially ordered sets, pp. 74-93 in Diskret. Mat. 1, 1989. (Russian; translation in Discrete Math. Appl. 1 (1991), no. 1, 35–58)
Follow up Questions	- How do induced matchings provide a natural lower bound for the number of maximal independent sets? - What role do stability results play in proving the upper bound for maximal independent sets in Q_n? - Can the methods in this paper be extended to other types of higher-dimensional graphs?

Article #20 Notes: Independent Sets in the Discrete Hypercube

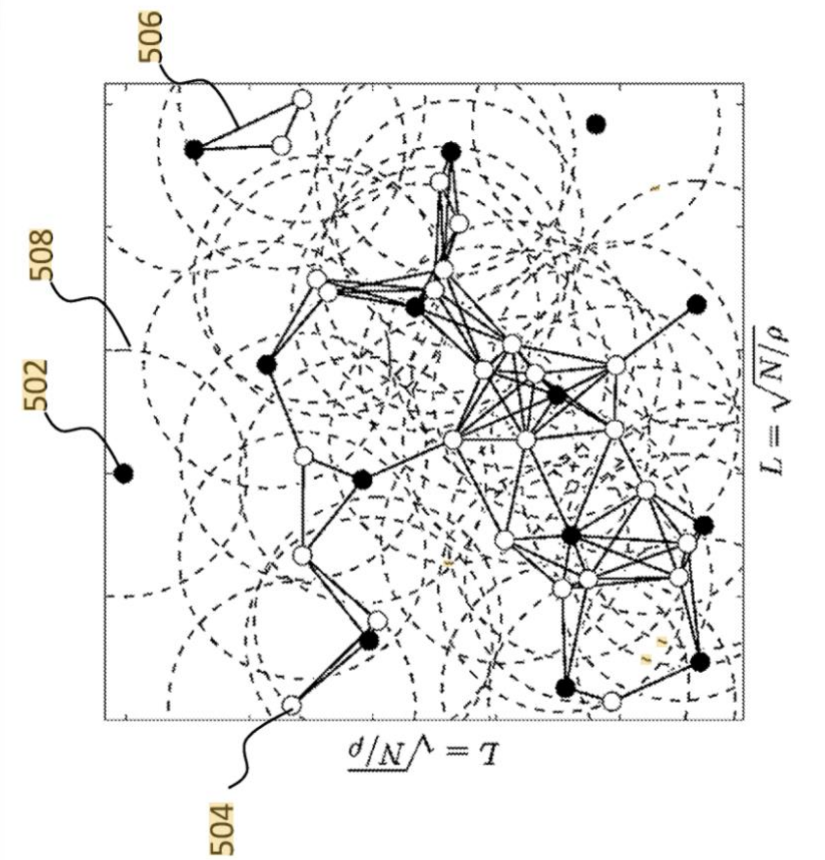
Source Title	Independent Sets in the Discrete Hypercube
Source citation (APA Format)	Galvin, D. (2019). Independent sets in the discrete hypercube. *arXiv preprint arXiv:1901.01991*.
Original URL	https://www3.nd.edu/~dgalvin1/pdf/countingindsetsinQd.pdf
Source type	Journal Article

Keywords	- Discrete hypercube - Independent sets - Asymptotic analysis - Graph theory - Bipartite graph
#Tags	#counting #algorithm #MIS
Summary of key points + notes (include methodology)	This paper discusses a result about the number of independent sets in the d-dimensional discrete hypercube (Q_d). The hypercube is a bipartite graph with two sets of vertices: those with an even number of 1's and those with an odd number. The paper presents a proof by A. Sapozhenko showing that the number of independent sets in Qd grows asymptotically as d tends to infinity. A basic lower bound on the number of independent sets is given as, and it is shown that this bound is close to the actual number of independent sets.
Research Question/Problem/ Need	What is the asymptotic behavior of the number of independent sets in the d-dimensional discrete hypercube, and how does this relate to the lower bound established by Korshunov and Sapozhenko?
Important Figures	$\lvert \mathcal{G}(a,g,v)\rvert \leq 2^{g-\frac{c't}{\log d}}.$ $\sum_{A \text{ of type II}} 2^{-\lvert N(A)\rvert} = \sum_{a\leq g,\ g\geq d^4,\ v} \lvert\mathcal{G}(a,g,v)\rvert 2^{-g} \leq \sum_{a\leq g,\ g\geq d^4,\ v} 2^{-\Omega\left(\frac{t}{\log d}\right)} \leq 2^{3d}2^{-\Omega\left(\frac{d^{7/2}}{\log d}\right)} = e^{-\Omega(d)}.$ $\lvert\mathcal{I}(Q_d)\rvert \sim 2\sqrt{e}2^{2^{d-1}} \qquad \text{as } d \to \infty.$
VOCAB: (w/definition)	Discrete hypercube (Q_d) - A graph where vertices represent binary strings of length d, with edges between vertices differing by exactly one bit.

	Asymptotic analysis - The study of the behavior of functions as the input grows large (in this case, as d tends to infinity) Bipartite graph - A graph whose vertices can be divided into two disjoint sets such that no two vertices within the same set are adjacent
Cited references to follow up on	L. Lov´asz, On the ratio of optimal integral and fractional covers, Discrete Math. 13 (1975), 383–390. B. Bollob´as, Random Graphs, Cambridge University Press, Cambridge, 2001. A. A. Sapozhenko, The number of antichains in ranked partially ordered sets, Diskret. Mat. 1 (1989), 74–93. (Russian; translation in Discrete Math. Appl. 1 (1991), 35–58)
Follow up Questions	How does the structure of the hypercube graph influence the number of independent sets it can have? Is there a way to, knowing the structure, intuitively reason why the bound is what it is? What role does the bipartite nature of the hypercube play in determining the bounds for the number of independent sets? Can the result about independent sets in the hypercube be generalized to hypercubes formed by k_n, or is it unique to the k_2 hypercube structure?

Patent #1: Quantum Computation for Combinatorial Optimization Problems Using Programmable Atomic Arrays

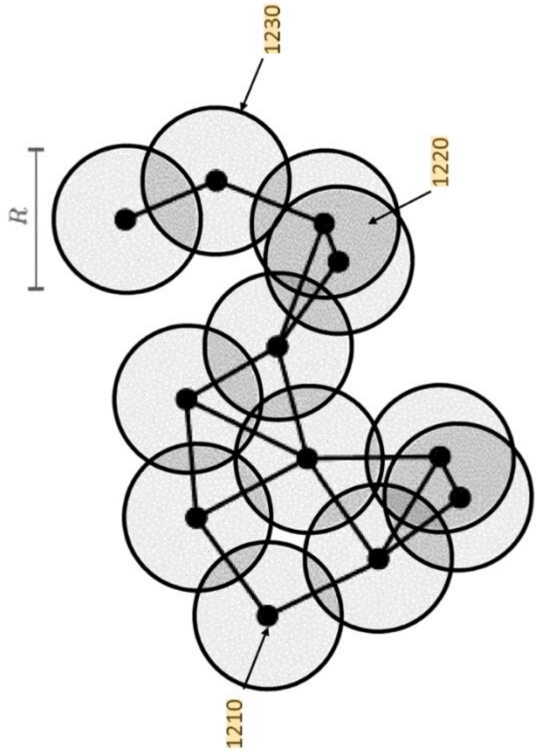
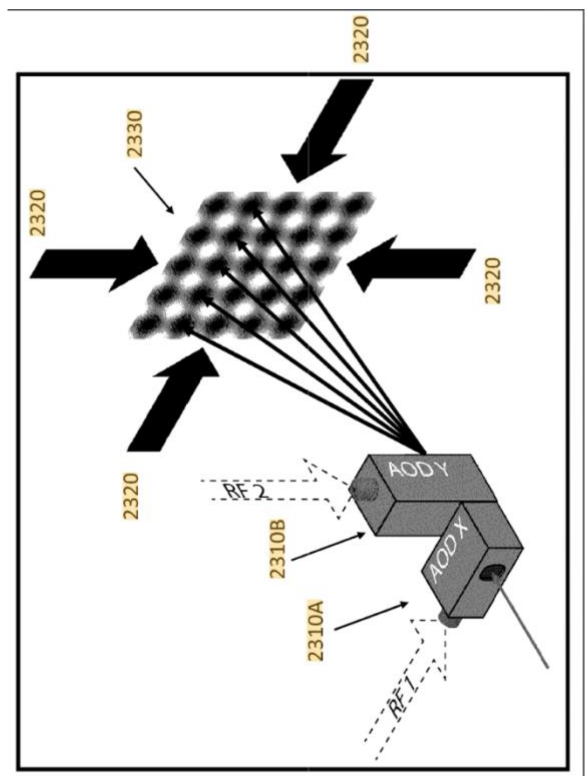
Source Title	Quantum Computation for Combinatorial Optimization Problems Using Programmable Atomic Arrays
Source citation (APA Format)	Pichler, H., Wang, S., Chou, L. X., Choi, S. W., & Lukin, M. D. (2024). *Quantum Computation for Combinatorial Optimization Problems Using Programmable Atomic Arrays. JP7546312B2*. Harvard College.
Original URL	https://patents.google.com/patent/JP7546312B2/
Source type	Patent
Keywords	NP Hard, Efficiency, Atomic Array, Nanotechnology

#Tags	#MIS #Counting
Summary of key points + notes (include methodology)	A quantum computer using atomic arrays can be used to find Maximum Independent Sets, a problem whose computational complexity would otherwise scale exponentially as the number of vertices increases. Neutral atoms can be used as 'building blocks' for large-scale quantum systems.
Research Question/Problem/ Need	How can someone find Maximum Independent Sets, an NP hard question, efficiently?
Important Figures	 FIG. 5
VOCAB: (w/definition)	Neutral Atom – an atom with no net electric charge Atomic Array – a data structure in where every element can be updated individually and completely
Cited references to follow up on	N/A?
Follow up Questions	What current algorithms are most efficient for finding Maximal Independent Sets? Despite being NP-hard, is the verification of an answer easier than deriving it?

	How can one construct an atomic array?

Patent #2: Neutral Atom Quantum Information Processor

Source Title	Neutral Atom Quantum Information Processor
Source citation (APA Format)	Kiesling Contreras, A., Bunyan, H., Schwartz, S., Levine, H. J., Omran, A., Lukin, M. D., Bultik, V., Endres, M., Greiner, M., Fischler, H., Zhou, L., Tao, K. S., Choi, S., Kim, D., & Ziprove, A. S. (2023). Maximal independent sets. *KR102609437B1*. Korean Intellectual Property Office. https://patents.google.com/patent/KR102609437B1/en
Original URL	https://patents.google.com/patent/KR102609437B1/en
Source type	Patent
Keywords	NP Hard, Efficiency, Atomic Array, Nanotechnology
#Tags	#MIS
Summary of key points + notes (include methodology)	Lazers can be used to arrange atoms into 1d and 2d arrays. Atoms are excited into Rydberg states, and can be used to solve NP Hard questions
Research Question/Problem/ Need	How can one construct an atomic array for use in finding Maximal Independent Sets?

Important Figures	 
VOCAB: (w/definition)	Rydberg states - electronically excited states that converge on an ionic state with an ionization energy

Cited references to follow up on	N/A?
Follow up Questions	Have atomic arrays been used before (+ for other things) in quantum computing? What makes some elements better choices for the array than others? Are atoms in the higher energy rydberg state unstable? How is the array preserved? (If it is)