Exploring Order-3 Maximal Independent Sets in Lattice-Torus Product Graphs

Grant Proposal

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Abstract

This project aims to count and investigate patterns in Order-3 Maximal Independent Sets (MIS) in graphs of the type $G = \left(\otimes_{i=1}^{a} P_{n_i} \right) \times \left(\otimes_{j=1}^{b} C_{m_j} \right) , \quad \forall\, i, j\ (n_i, m_j > 1)$, which will be referred to as lattice-torus product graphs. The 'Order-3' classification of MIS provides a measure of packing optimality to distinguish MIS by finding sets that cannot be enlarged by the addition and removal of 3 vertices. This research will use computational and combinatorial methods to explore the number of these sets in Lattice-Torus Product Graphs and would hopefully contribute to graph theory by helping to optimize the existing applications of MIS, like network design, scheduling, and computation theory.

Keywords: Independent Set, Order-k Maximal, Cartesian Product, Lattice, Torus

Exploring Order-3 Maximal Independent Sets in Lattice-Torus Product Graphs

In graph theory, an independent set is a subset of vertices in a graph such that no two vertices in the set are connected by an edge. These structures are extremely useful in understanding combinatorial properties of graphs and have practical applications in areas such as network design and scheduling (van Bevern et al., 2012), as well as coding theory (Bachoc, n.d.). A maximal independent set is an independent set that cannot be enlarged by adding another vertex without violating the independence condition. Maximal Independent Sets need to be both sparse to satisfy independence and dense to satisfy maximality, leading to natural questions about the bounds of acceptable densities. This project aims to find the size of these goldilocks zones for different sizes and structures of graphs by counting the number of Order-3 Maximal Independent Sets on Lattice-Torus Product Graphs, which is expected to reveal linear recurrence sequences. Preliminary data suggests that $C_3 \times C_n$ follows the recurrence relation of s(3)=6, s(4)=18, s(n) = 2*s(n-2) + s(n-1), while sets on larger graphs are described by more complicated sequences of higher order.

Order-k Maximal Independent Sets

Building upon the concept of maximal independent sets, order-k maximal independent sets, first studied by Yanco and Bagchi, generalize maximality by creating classes of packing optimality (Yanco & Bagchi, 1994). Specifically, an Order-k maximal independent set, for odd integer k, is defined as one that cannot be transformed into another larger independent set by adding or removing k vertices from the set. This generalization creates a metric to quantify how optimal the packing is by measuring the stability of independent sets under 'disruptions' of different sizes. Sets that aren't very tightly packed can fit additional vertices under small disruptions, while sets that are very tightly packed would need much larger disruptions to fit more vertices.

Yanco and Bagchi discovered and proved that the number of order-k maximal independent sets on cycle graphs is always a multiple of the number of vertices if the number of vertices is prime. For order-3 maximal independent sets, for example, this can be seen by verifying that the rightmost column is a multiple of the leftmost column for prime cycle lengths.

TABLE I

Cycle Length	Number of Colored Vertices																	$\|XC_{3,n}\|$
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	
1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	0
2	2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	2
3	3	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3
4	-	2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	2
5	-	5	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	5
6	-	-	2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	2
7	-	-	7	-	-	-	-	-	-	-	-	-	-	-	-	-	-	7
8	-	-	-	2	-	-	-	-	-	-	-	-	-	-	-	-	-	2
9	-	-	-	9	-	-	-	-	-	-	-	-	-	-	-	-	-	9
10	-	-	-	5	2	-	-	-	-	-	-	-	-	-	-	-	-	7
11	-	-	-	-	11	-	-	-	-	-	-	-	-	-	-	-	-	11
12	-	-	-	-	12	2	-	-	-	-	-	-	-	-	-	-	-	14
13	-	-	-	-	-	13	-	-	-	-	-	-	-	-	-	-	-	13
14	-	-	-	-	-	21	2	-	-	-	-	-	-	-	-	-	-	23
15	-	-	-	-	-	5	15	-	-	-	-	-	-	-	-	-	-	20
16	-	-	-	-	-	-	32	2	-	-	-	-	-	-	-	-	-	34
17	-	-	-	-	-	-	17	17	-	-	-	-	-	-	-	-	-	34
18	-	-	-	-	-	-	-	45	2	-	-	-	-	-	-	-	-	47
19	-	-	-	-	-	-	-	38	19	-	-	-	-	-	-	-	-	57
20	-	-	-	-	-	-	-	5	60	2	-	-	-	-	-	-	-	67
21	-	-	-	-	-	-	-	-	70	21	-	-	-	-	-	-	-	91
22	-	-	-	-	-	-	-	-	22	77	2	-	-	-	-	-	-	101
23	-	-	-	-	-	-	-	-	-	115	23	-	-	-	-	-	-	138
24	-	-	-	-	-	-	-	-	-	60	96	2	-	-	-	-	-	158
25	-	-	-	-	-	-	-	-	-	5	175	25	-	-	-	-	-	205
26	-	-	-	-	-	-	-	-	-	-	130	117	2	-	-	-	-	247
27	-	-	-	-	-	-	-	-	-	-	27	252	27	-	-	-	-	306
28	-	-	-	-	-	-	-	-	-	-	-	245	140	2	-	-	-	387
29	-	-	-	-	-	-	-	-	-	-	-	87	348	29	-	-	-	464
30	-	-	-	-	-	-	-	-	-	-	-	5	420	165	2	-	-	592
31	-	-	-	-	-	-	-	-	-	-	-	-	217	465	31	-	-	713
32	-	-	-	-	-	-	-	-	-	-	-	-	32	672	192	2	-	898
33	-	-	-	-	-	-	-	-	-	-	-	-	-	462	605	33	-	1100
34	-	-	-	-	-	-	-	-	-	-	-	-	-	119	1020	221	2	1362
35	-	-	-	-	-	-	-	-	-	-	-	-	-	5	882	770	35	1692
36	-	-	-	-	-	-	-	-	-	-	-	-	-	-	336	1485	252	2075

This table presents data on order-3 MIS in cycle graphs. The table is comprised of 36 rows and 17 columns. Each row corresponds to a cycle graph of a specific size (the cycle length), and each column represents the number of colored vertices per set. Every cell shows the number of maximal independent sets for that given combination. In the rightmost column, each row's values are summed, and a distinct pattern emerges: For graphs with a prime cycle length, this sum is a multiple of the cycle length. This table is from (Yanco & Bagchi, 1994).

This pattern, however, does not hold for path graphs. For order-3 maximal independent sets, for example, this can be seen by observing that the rightmost column is not a multiple of the leftmost column for all prime cycle lengths.

TABLE II

Path Length	Number of Colored Vertices																	$\|XP_{3,n}\|$
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	
1	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1
2	2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	2
3	-	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1
4	-	3	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	3
5	-	1	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	2
6	-	-	4	-	-	-	-	-	-	-	-	-	-	-	-	-	-	4
7	-	-	3	1	-	-	-	-	-	-	-	-	-	-	-	-	-	4
8	-	-	-	5	-	-	-	-	-	-	-	-	-	-	-	-	-	5
9	-	-	-	6	1	-	-	-	-	-	-	-	-	-	-	-	-	7
10	-	-	-	1	6	-	-	-	-	-	-	-	-	-	-	-	-	7
11	-	-	-	-	10	1	-	-	-	-	-	-	-	-	-	-	-	11
12	-	-	-	-	4	7	-	-	-	-	-	-	-	-	-	-	-	11
13	-	-	-	-	-	15	1	-	-	-	-	-	-	-	-	-	-	16
14	-	-	-	-	-	10	8	-	-	-	-	-	-	-	-	-	-	18
15	-	-	-	-	-	1	21	1	-	-	-	-	-	-	-	-	-	23
16	-	-	-	-	-	-	20	9	-	-	-	-	-	-	-	-	-	29
17	-	-	-	-	-	-	5	28	1	-	-	-	-	-	-	-	-	34
18	-	-	-	-	-	-	-	35	10	-	-	-	-	-	-	-	-	45
19	-	-	-	-	-	-	-	15	36	1	-	-	-	-	-	-	-	52
20	-	-	-	-	-	-	-	1	56	11	-	-	-	-	-	-	-	68
21	-	-	-	-	-	-	-	-	35	45	1	-	-	-	-	-	-	81
22	-	-	-	-	-	-	-	-	6	84	12	-	-	-	-	-	-	102
23	-	-	-	-	-	-	-	-	-	70	55	1	-	-	-	-	-	126
24	-	-	-	-	-	-	-	-	-	21	120	13	-	-	-	-	-	154
25	-	-	-	-	-	-	-	-	-	1	126	66	1	-	-	-	-	194
26	-	-	-	-	-	-	-	-	-	-	56	165	14	-	-	-	-	235
27	-	-	-	-	-	-	-	-	-	-	7	210	78	1	-	-	-	296
28	-	-	-	-	-	-	-	-	-	-	-	126	220	15	-	-	-	361
29	-	-	-	-	-	-	-	-	-	-	-	28	330	91	1	-	-	450
30	-	-	-	-	-	-	-	-	-	-	-	1	252	286	16	-	-	555
31	-	-	-	-	-	-	-	-	-	-	-	-	84	495	105	1	-	685
32	-	-	-	-	-	-	-	-	-	-	-	-	8	462	364	17	-	851
33	-	-	-	-	-	-	-	-	-	-	-	-	-	210	715	120	1	1046
34	-	-	-	-	-	-	-	-	-	-	-	-	-	36	792	455	18	1301
35	-	-	-	-	-	-	-	-	-	-	-	-	-	1	462	1001	136	1601
36	-	-	-	-	-	-	-	-	-	-	-	-	-	-	120	1287	560	1986

This table presents data on order-3 maximal independent sets in path graphs. Unlike the cyclic case, the property observed in cycle graphs — where the total number of sets (the sum in the rightmost column) is a multiple of the cycle length for prime lengths — does not hold for paths. This table is from (Yanco & Bagchi, 1994).

No pseudoprimes were found for order-3 maximal independent sets on cycles, but some were found for other orders. For order-1 maximal independent sets, the sequence was shown to be equivalent to the Perrin Sequence, and, thus, 271441 was the first pseudoprime.

Lattice-Torus Product Graphs

The number of Order-k maximal independent sets have thus far only been counted on the graphs P_n and C_n , referring to path and cycle graphs of n vertices, respectively. The research on cycle graphs revealed symmetric properties that were used to generate families of sequences such that in any of the sequences a , $p|a_p$ for all prime – and a few pseudoprime – p (Yanco & Bagchi, 1994). No further research has been done on the properties or on order-k maximal independent sets on different graph types. For this reason, we define a new type of graph that is created by taking the cross product of a (possibly zero) number of path and cycle graphs and analyze the number of order-k maximal independent sets on them. We define a Lattice-Torus Product Graph as a graph of the form

$$G = \left(\otimes_{i=1}^{a} P_{n_i} \right) \times \left(\otimes_{j=1}^{b} C_{m_j} \right) \quad \text{s.t. } \forall\, i, j, \ \ (n_i, m_j > 1) \ \& \ a, b \in \mathbb{Z}_{\geq 0} \,. \tag{1}$$

The graphs described in (1) are formed by the cartesian product of an a -dimensional lattice graph and a b -dimensional tori graph. For $(a, b) = (1,0)$ or $(0,1)$, the graphs are simply paths or cycles, but counting the generalized combination of these structures will hopefully reveal new patterns, sequences, and properties, that extend the previous work by (Yanco & Bagchi, 1994) on order-k maximal independent sets to a broader class of graphs. Other than counting the number of order-3 maximal independent sets, this project also aims to find bounds on the smallest order-3 maximal independent set for different graph sizes, essentially calculating the minimum density needed to maintain a certain threshold of optimality. Preliminary data of order-3 maximal independent sets on Lattice, Cylinder, and Torus graphs has yielded data worth further investigation. Below, tables for Lattice, Cylinder, and Torus graphs are shown:

TABLE III

		Path 1 Length											
		2	3	4	5	6	7	8	9	10	11	12	13
	2	2	2	2	4	6	8	12	18	26	38	56	82
	3	2	2	6	10	14	25	44	71	118	201	336	
	4	2	6	14	28	44	88	178	344	632	1244		
	5	4	10	28	64	118	265	622	1371	2966			
	6	6	14	44	118	256	682	1846	4612				
	7	8	25	88	265	682	2168	6704					
	8	12	44	178	622	1846	6704						
	9	18	71	344	1371	4612							
Path 2 Length	10	26	118	632	2966								
	11	38	201	1244									
	12	56	336										
	13	82											

This table presents the number of Order-3 Maximal Independent Sets on Lattice graphs. Each row and column describes the length of one Path graph contained in the Lattice graph.

TABLE IV

		Path Length									
		2	3	4	5	6	7	8	9	10	11
	2										
	3	3	12	24	48	96	192	384	768	1536	3072
	4	2	4	10	16	32	66	122	238	468	
	5	10	20	40	90	200	450	1010	2270		
	6	8	22	54	152	408	1128	2996			
	7	14	28	70	196	518	1428				
	8	26	92	338	1088	3680					
	9	24	66	222	1002	3678					
Cycle Length	10	52	194	836	3368						
	11	66	330	1738							
	12	92	382								
	13	156									

This table presents the number of Order-3 Maximal Independent Sets on Cylinder graphs. Each row describes the length of the Path graph contained in the Cylinder graph, while each column describes the length of the Cycle graph contained in it.

TABLE V

		Cycle 1 Length										
		2	3	4	5	6	7	8	9	10	11	12
	2											
	3		6	18	30	66	126	258	510	1026	2046	4098
	4		18	2	30	80	42	250	432	532	1782	
	5		30	30	60	190	280	710	1560	3340		
	6		66	80	190	920	980	5120	11856			
	7		126	42	280	980	1316					
Cycle 2 Length	8		258	250	710	5120						
	9		510	432	1560	11856						
	10		1026	532	3340							
	11		2046	1782								
	12		4098									

This table presents the number of Order-3 Maximal Independent Sets on Torus graphs. Each row and column describe the length of one Cycle graph contained in the torus.

Section II: Specific Aims & Objectives

This proposal's objective is to investigate Order-k Maximal Independent Sets on a variety of Lattice-Torus Product Graphs. My long-term goal is to contribute to the understanding of packing optimality and stability in sets, by extending the current research on Order-k maximal independent sets to new graph types. The central hypothesis of this proposal is that the study of these sets on more complex graph types will reveal sequences with interesting properties. Specifically, I hypothesize that for all Lattice-Torus Product Graphs, the number of order-k maximal independent sets will be a multiple of the product of the prime elements of

$\{m_1, m_2, \ldots m_b\}$.

Hypothesis Rationale

The rationale behind this hypothesis is that traditional cycles graphs have been shown to have symmetries when Order-k maximal independent sets are counted on them (Yanco & Bagchi, 1994). Thus, when examining torus graphs and graphs with a toroidal component comprised of cycle graphs, I would expect to find similar structures and properties. I believe that the rotation of the set about a certain cycle component would reveal a rotational symmetry that would cause the total number of order-k maximal independent sets to be a multiple of that number for cycles with a prime number of vertices.

The work will focus on three specific aims:

Development of an Algorithm for Counting Order-k Maximal Independent Sets on General Graphs

A method to identify and count Order-k maximal independent sets will be developed for general graphs. Additional combinatorial approaches might be optimized to leverage properties of Lattice-Torus Product Graphs. The algorithm should be able to identify a maximal independent set as Order-k maximal in O(n^(k-1)/2) time, where n is the number of vertices in the graph. However, more optimized approaches are likely possible for specific graph classes. The algorithm will then be implemented with Python on Lattice-Torus Product Graphs to obtain a dataset.

Development of an equation relating the Number of Order-k Maximal Independent Sets on Lattice-Torus Product Graphs

An expression for counting Order-k maximal independent sets will be developed. Although a closed form would be ideal, relationships between the number of order-k maximal independent sets on different graph types would also be acceptable. Thus far, relational expressions and closed forms have been developed for fundamental path and cycle graphs (Yanco & Bagchi, 1994). By leveraging their unique structure, I hope to be able to generalize these results to more complex product graphs. The resulting formula should provide a method to calculate the number of Order-k maximal independent sets without requiring as much computational power as brute-force counting, thus allowing for more pattern recognition in the data.

Determination of Bounds on the Smallest Order-k Maximal Independent Set

I will use the data collected to hypothesize bounds for the smallest Order-k maximal independent set in Lattice-Torus Product Graphs. By analyzing the structure and properties of these graphs, I will calculate the minimum density of vertices required to maintain the threshold level of optimality required for the Order-k criterion. These bounds should provide insight into the required density of points needed to have order-k maximality.

The expected outcome of this work is a deeper understanding of Order-k maximal independent sets. I anticipate finding patterns in the sequences generated and hope to develop methods that would contribute to both the study of graphs and their practical applications in optimization. Additionally, this work will provide a foundation for future research exploring other classes of graphs and related problems.

Section III: Resources/Equipment

This project requires both high-performance hardware and software. The resources below provide a significant amount of computational power to find Order-k maximal independent sets, while keeping costs reasonable.

Hardware

The primary computational resource for this project would be a system powered by Threadripper processors. These processors are equipped with a high number of cores and multithreading capabilities, making them ideal for tasks like counting large numbers of order-k maximal independent sets.

Software

I plan to use Python and its libraries to code the software. Python is particularly well-suited for this project due to its ease of use and wide array of mathematical and graph-theoretic tools. I'm planning to use the following Python libraries:

NetworkX

This library provides tools for constructing and analyzing complex graphs, and I think I could use it to generate Lattice-Torus Product Graphs.

NumPy and SciPy

These libraries are helpful for handling large datasets and performing advanced numerical computations. They will efficiently store and manipulate data generated during the project. They can also be used to manipulate large arrays, which could be useful for storing sets and graphs.

SymPy

When deriving closed-form expressions for the number of Order-k maximal independent sets, SymPy's symbolic computation will help combine algebraic relationships into closed forms and calculate data to verify hypothesized relationships against my calculated results.

Section V: Ethical Considerations

Since this project doesn’t involve human participants, animal subjects, or any form of environmental impact, there aren’t any ethical concerns or risks that need to be considered.

Section VIII: References

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