Checking Solvability of Systems of Interval Linear Equations and Inequalities via Mixed Integer Programming*

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Abstract

This paper deals with the problems of checking strong solvability and feasibility of linear interval equations, checking weak solvability of linear interval equations and inequalities, and finding control solutions of linear interval equations. These problems are known to be NP-hard. We use some recently developed characterizations in combination with classical arguments to show that these problems can be equivalently stated as optimization tasks and provide the corresponding linear mixed 0–1 programming formulations.

Keywords: interval linear systems; mixed integer programming; solvability; Farkas lemma

1 Introduction

Systems of linear interval equations and inequalities frequently arise in practice, especially in situations where the data cannot be measured exactly but are known to be in a certain range [2,6]. In this paper, we consider the problems of checking strong solvability and feasibility of linear interval equations; checking weak solvability of linear interval equations and inequalities; and finding control solutions of linear interval equations. It is known that these problems are NP-hard [7, 8, 13, 14, 16]. The reader is referred to the classical book [3] for background on computational complexity theory.

Several enumeration algorithms for solving these problems have been proposed in the literature [16]. In this paper we show that these problems can be equivalently represented as linear mixed 0–1 programs, and thus can be alternatively solved using well-established mixed integer programming solvers (e.g., Xpress-MP [1] or CPLEX [5]).

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Throughout the paper, we follow the definitions and notations given in [16], which provides a detailed survey on solvability of systems of interval linear equations and inequalities.

Let $\underline{A}$ and $\overline{A}$ be two matrices in $\mathbb{R}^{m\times n}$ and $\underline{A} \le \overline{A}$. The set of matrices

$$\mathbf{A} = [\underline{A}, \overline{A}] = \{A \mid \underline{A} \le A \le \overline{A}\} \tag{1}$$

is called an interval matrix, while matrices $\underline{A}$ and $\overline{A}$ are referred to as its lower and upper bounds. If $\underline{A} = (\underline{a}_{ij})$ and $\overline{A} = (\overline{a}_{ij})$ then $\mathbf{A}$ is the set of matrices $A = (a_{ij})$ such that for all i and j

$$\underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}. \tag{2}$$

Similarly, we define an interval vector $\mathbf{b}$ as a one-column interval matrix

$$\mathbf{b} = \{b \mid \underline{b} \le b \le \overline{b}\}, \tag{3}$$

where $\underline{b}, \overline{b} \in \mathbb{R}^m$.

A system of interval linear equations $\mathbf{A}x = \mathbf{b}$ is defined as a family of all systems of linear equations $Ax = b$, where $A \in \mathbf{A}$ and $b \in \mathbf{b}$. Following the same idea, a system of interval linear inequalities $\mathbf{A}x \le \mathbf{b}$ is defined as a family of all systems of linear inequalities $Ax \le b$ with data satisfying $A \in \mathbf{A}$ and $b \in \mathbf{b}$.

A system of linear equations $Ax = b$ is called *solvable* if it has a solution, and *feasible* if it has a nonnegative solution. A system of linear interval equations $\mathbf{A}x = \mathbf{b}$ is called *weakly solvable (feasible)* if there exists $A \in \mathbf{A}$ and $b \in \mathbf{b}$ such that system $Ax = b$ is solvable (feasible), and it is called *strongly solvable (feasible)* if the system $Ax = b$ is solvable (feasible) for all $A \in \mathbf{A}$ and $b \in \mathbf{b}$. In the same manner we define the weak and strong solvability (feasibility) of a system of linear interval inequalities $\mathbf{A}x \le \mathbf{b}$.

A vector $x \in \mathbb{R}^n$ is called a *weak solution* of $\mathbf{A}x = \mathbf{b}$ ($\mathbf{A}x \le \mathbf{b}$) if it satisfies $Ax = b$ ($Ax \le b$) for some $A \in \mathbf{A}$ and $b \in \mathbf{b}$. A vector $x \in \mathbb{R}^n$ is called a *control solution* of $\mathbf{A}x = \mathbf{b}$ if for each $b \in \mathbf{b}$ there exists a matrix $A \in \mathbf{A}$ such that $Ax = b$ holds, or, equivalently,

$$\mathbf{b} \subseteq \{Ax \mid A \in \mathbf{A}\}. \tag{4}$$

Next, we introduce some additional notations used. Denote by A_c and Δ the center and radius matrices given by

$$A_c = \frac{1}{2}(\overline{A} + \underline{A}) \quad \text{and} \quad \Delta = \frac{1}{2}(\overline{A} - \underline{A}), \tag{5}$$

respectively. Similarly, the center and radius vectors are defined as

$$b_c = \frac{1}{2}(\overline{b} + \underline{b}) \quad \text{and} \quad \delta = \frac{1}{2}(\overline{b} - \underline{b}), \tag{6}$$

respectively. Let Y_m be the set of all $\{-1, 1\}$ m-dimensional vectors, i.e.,

$$Y_m = \{y \in \mathbb{R}^m \mid |y| = e\}, \tag{7}$$

where $e = (1, \ldots, 1)^T$ is the m-dimensional vector of all 1's. For a given $y \in Y_m$, let

$$T_y = \mathrm{diag}(y_1, \ldots, y_m) \tag{8}$$

denote the corresponding diagonal matrix. For a given interval matrix $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ and vectors $y \in Y_m$ and $z \in Y_n$, we introduce the following matrices:

$$A_{yz} = A_c - T_y \Delta T_z. \tag{9}$$

Similarly, for an interval vector $\mathbf{b} = [b_c - \delta, b_c + \delta]$ and $y \in Y_m$, we define vectors

$$b_y = b_c + T_y \delta. \tag{10}$$

The remainder of this paper is organized as follows. Section 2 provides some preliminary results used in our proofs. Sections 3 and 4 deal with the problem of checking strong solvability and feasibility of linear interval equations and checking weak solvability of systems of linear interval equations and inequalities, respectively. The problem of finding control solutions of a system of linear interval equations is addressed in Section 5. Finally, Section 6 concludes the paper.

2 Preliminaries

The following result by Rohn [15] characterizes strong solvability of systems of linear equations.

Theorem 1 (Rohn) *A system $\boldsymbol{A}x = \boldsymbol{b}$ is strongly solvable if and only if for each $y \in Y_m$ the system*

$$A_{ye}x^1 - A_{-ye}x^2 = b_y, \tag{11}$$

$$x^1 \geq 0, \ x^2 \geq 0, \tag{12}$$

has a solution x_y^1, x_y^2. Moreover, if this is the case, then for each $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$ the system $Ax = b$ has a solution in the set

$$Conv\{x_y^1 - x_y^2 \mid y \in Y_m\}.$$

The above result implies that we can check strong solvability of $\mathbf{A}x = \mathbf{b}$ via a simple enumeration procedure, which requires solving a linear system (11)-(12) for each $y \in Y_m$. In the worst case scenario the number of systems necessary to be checked is $O(2^m)$. The details of the algorithm as well as some clever adjustments to improve its practical performance are discussed in [16].

Characterization of strong feasibility can be derived from strong solvability conditions (see [16] for details).

Theorem 2 (Rohn) *A system $\boldsymbol{A}x = \boldsymbol{b}$ is strongly feasible if and only if for each $y \in Y_m$ the system*

$$A_{ye}x = b_y, \tag{13}$$

has a nonnegative solution x_y. Moreover, if this is the case, then for each $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$ the system $Ax = b$ has a solution in the set

$$Conv\{x_y \mid y \in Y_m\}.$$

The following key result dealing with checking weak solvability of equations is proved in [9]:

Theorem 3 (Oettli-Prager) *A vector $x \in \mathbb{R}^n$ is a weak solution of $\boldsymbol{A}x = \boldsymbol{b}$ if and only if it satisfies*

$$|A_c x - b_c| \leq \Delta|x| + \delta. \tag{14}$$

An equivalent statement is provided in [16]:

Theorem 4 (Rohn) *A system $\boldsymbol{A}x = \boldsymbol{b}$ is weakly solvable if and only if the system*

$$A_{ez}x \leq \overline{b} \tag{15}$$

$$-A_{-ez}x \leq -\underline{b} \tag{16}$$

is solvable for some $z \in Y_n$.

The above result implies that we can check weak solvability of $\mathbf{A}x = \mathbf{b}$ via a simple enumeration procedure, which requires for each $z \in Y_n$ to solve a respective linear system (15)-(16) (see [16] for discussion).

An analogue of the Oettli-Prager theorem for the case of inequalities was proved by Gerlach [4, 16].

Theorem 5 (Gerlach) *A vector $x \in \mathbb{R}^n$ is a weak solution of $\boldsymbol{A}x \leq \boldsymbol{b}$ if and only if it satisfies*

$$A_c x - \Delta|x| \leq \overline{b}. \tag{17}$$

An equivalent statement is provided in [16]:

Theorem 6 (Rohn) *A system $\boldsymbol{A}x \leq \boldsymbol{b}$ is weakly solvable if and only if the system*

$$A_{ez}x \leq \overline{b} \tag{18}$$

is solvable for some $z \in Y_n$.

The characterizations of control solutions were introduced in [17]. It is known that (see [16]):

Theorem 7 (Lakeyev-Noskov, Rohn) *The following assertions are equivalent:*

(i) x is a control solution of $\boldsymbol{A}x = \boldsymbol{b}$.

(ii) *x satisfies*

$$|A_c x - b_c| \le \Delta|x| - \delta. \tag{19}$$

(iii) *x solves*

$$A_{ez}x \le \underline{b} \tag{20}$$

$$-A_{-ez}x \le -\overline{b} \tag{21}$$

for some $z \in Y_n$.

In the proofs, we will also use the classical Farkas lemma (see, e.g., [12]), which we state for the sake of completeness.

Lemma 1 (Farkas) *A system*

$$Ax = b, \;\; x \ge 0$$

has no solution if and only if the system

$$A^T p \ge 0, \;\; b^T p < 0 \tag{22}$$

has a solution.

3 Strong solvability and feasibility of a system of equations

This section formulates the problems of checking the strong solvability and strong feasibility of systems of interval linear equations as equivalent mixed 0-1 programming problems. We treat the solvability and feasibility in two separate subsections.

3.1 Checking strong solvability

Consider the following linear mixed 0–1 programming problem:

$$\max_{\alpha,w,p,z} \;\; \alpha \tag{23}$$

subject to

$$-\Delta^T w \le A_c^T p \le \Delta^T w, \tag{23a}$$

$$b_c^T p - \delta^T w \le -\alpha, \tag{23b}$$

$$0 \le w + p \le e - z, \tag{23c}$$

$$0 \le w - p \le z, \tag{23d}$$

$$\alpha \ge 0, \;\; z \in \{0,1\}^m, \tag{23e}$$

$$\alpha \in \mathbb{R}^1, w, p \in \mathbb{R}^m. \tag{23f}$$

Proposition 1 *A system* $\boldsymbol{A}x = \boldsymbol{b}$ *is strongly solvable if and only if* $\alpha^* = 0$ *in any optimal solution of (23).*

Proof. The proof is based on applying Theorem 1 and Farkas lemma. First, observe that the mixed linear 0–1 program (23) is always feasible since $\alpha = 0$, $p = w = 0$ and any 0–1 vector z satisfy the constraints in (23). Also, note that from inequalities (23c) and (23d) it follows that

$$w = |p| \tag{24}$$

and

$$\|p\|_\infty = \|w\|_\infty \leq 1/2. \tag{25}$$

From (23b) and (25) we have

$$\alpha \leq |b_c^T p| + |\delta^T w| \leq \frac{1}{2}(\|b_c\|_1 + \|\delta\|_1), \tag{26}$$

meaning that the objective value of (23) is bounded. Thus, the program (23) has a finite optimal solution with $0 \leq \alpha < \infty$.

Let α^* be the optimal objective value of (23). We can prove the theorem by showing that

(i) if $\alpha^* > 0$ then $\mathbf{A}x = \mathbf{b}$ is not strongly solvable;

(ii) if $\mathbf{A}x = \mathbf{b}$ is not strongly solvable then $\alpha^* > 0$.

First, we prove (i). For each $i \in \{1, \ldots, m\}$ we define y_i as follows:

$$y_i = \begin{cases} -1, & \text{if } p_i \geq 0, \\ 1, & \text{if } p_i < 0. \end{cases}$$

Due to (24), we can rewrite w as $w = -T_y p$, where T_y is as defined in (8).

Since $\alpha^* > 0$, the constraints (23a) and (23b) imply the existence of $p \in \mathbb{R}^m$ such that

$$\Delta^T T_y p \leq A_c^T p \leq -\Delta^T T_y p \tag{27}$$

and

$$b_c^T p + \delta^T T_y p < 0, \tag{28}$$

respectively.

Using the definition of matrices A_{ye}, A_{-ye} and vector b_y according to (9)-(10), the inequalities (27)-(28) can be equivalently rewritten as

$$A_{ye}^T p \geq 0, \tag{29}$$

$$A_{-ye}^T p \leq 0, \tag{30}$$

$$b_y^T p < 0. \tag{31}$$

Denote by

$$\check{A} = [A_{ye}; -A_{-ye}] \tag{32}$$

the $m \times (2n)$ matrix obtained by juxtaposing the matrices A_{ye} and $-A_{-ye}$. Then the inequalities (29)-(30) are equivalent to

$$\breve{A}^T p \geq 0, \tag{33}$$

and according to Farkas lemma applied to $\breve{A}$ and b_y, inequalities (31) and (33) represent sufficient conditions for the system

$$\breve{A}x = b_y,\ x \geq 0$$

not to have a solution $x \in \mathbb{R}^{2n}$. Denoting by $x^1 = [x_1, \dots, x_n]^T$ and $x^2 = [x_{n+1}, \dots, x_{2n}]^T$, the last system becomes exactly (11)-(12) for a particular $y \in Y_m$. Applying Theorem 1 completes the proof of part (i).

Next, we prove part (ii). Assume that $\mathbf{A}x = \mathbf{b}$ is not strongly solvable. Then by Theorem 1 for some $y \in Y_m$ system (11)-(12) does not have a solution. Therefore, by Farkas lemma applied to $\breve{A}$ (as defined in (32)) and b_y, inequalities (31) and (33), and hence (29)-(31), should be satisfied for some $\tilde{p}$. Since the right-hand sides in (29)-(31) are zeros, multiplying $\tilde{p}$ by a suitably small positive constant we can obtain p, which satisfies (29)-(31) and $\|p\|_\infty \leq 1/2$. Hence, by definition of matrices A_{ye}, A_{-ye} and vector b_y, there exists $\alpha > 0$ such that

$$\Delta^T T_y p \leq A_c^T p \leq -\Delta^T T_y p, \tag{34}$$

$$b_c^T p + \delta^T T_y p \leq -\alpha. \tag{35}$$

Next, for each $i \in \{1, \dots, m\}$, we define $\breve{y}_i$ as follows:

$$\breve{y}_i = \begin{cases} -1, & \text{if } p_i \geq 0, \\ 1, & \text{if } p_i < 0. \end{cases}$$

Note that since $\delta \geq 0$ and $\Delta \geq 0$, inequalities (34)-(35) remain valid if y is replaced with $\breve{y}$. If we define $w = -T_{\breve{y}} p$, then $w = |p|$ and $\|p\|_\infty = \|w\|_\infty \leq 1/2$, which implies that

$$\begin{array}{l} 0 \leq w + p \leq e - z, \\ 0 \leq w - p \leq z, \end{array} \tag{36}$$

for vector $z \in \{0,1\}^m$ such that $z_i = 1$ if $p_i < 0$ and $z_i = 0$ otherwise, $i = 1, \dots, m$. Therefore, there exists a feasible solution for (23) with $\alpha > 0$, and due to (26), the problem (23) has a positive optimal objective value α^*. ∎

3.2 Checking strong feasibility

Consider the following linear mixed 0–1 programming problem:

$$\max_{\alpha, w, p, z} \alpha \tag{37}$$

subject to

$$A_c^T p \geq -\Delta^T w, \tag{37a}$$

$$b_c^T p - \delta^T w \leq -\alpha, \tag{37b}$$

$$0 \leq w + p \leq e - z, \tag{37c}$$

$$0 \leq w - p \leq z, \tag{37d}$$

$$\alpha \geq 0, \ z \in \{0,1\}^m, \tag{37e}$$

$$\alpha \in \mathbb{R}^1, w, p \in \mathbb{R}^m. \tag{37f}$$

Proposition 2 *A system* $\boldsymbol{A}x = \boldsymbol{b}$ *is strongly feasible if and only if* $\alpha^* = 0$ *in any optimal solution of (37).*

Proof. The proof is based on applying Theorem 2 and Farkas lemma and is analogous to the proof of Proposition 1. It is left as an exercise for the reader. ■

4 Weak solvability of systems of equations and inequalities

This section consists of two subsections addressing the weak solvability of systems of linear interval equations and inequalities, respectively.

4.1 Checking weak solvability of a system of equations

For systems of linear interval equations, we will provide two equivalent characterizations of their weak solvability, both utilizing Theorem 3.

Observe that (14) can be equivalently rewritten as

$$A_c x - b_c \leq \Delta|x| + \delta \tag{38}$$

$$-(A_c x - b_c) \leq \Delta|x| + \delta. \tag{39}$$

Let us introduce for a moment some large constant β as an upper bound on x_i for all i, i.e.,

$$\|x\|_\infty \leq \beta, \tag{40}$$

and replace $|x|$ with a new variable y:

$$y = |x|.$$

Then the last expression can be equivalently written as

$$0 \leq y + x \leq 2\beta z, \ 0 \leq y - x \leq 2\beta(e - z), \ z \in \{0,1\}^n.$$

Then the system consisting of (14) and (40) is equivalent to the following mixed 0–1 feasibility problem:

$$-(\Delta y + \delta) \le A_c x - b_c \le \Delta y + \delta, \tag{41}$$

$$0 \le y + x \le 2\beta z, \tag{42}$$

$$0 \le y - x \le 2\beta(e - z), \tag{43}$$

$$x, y \in \mathbb{R}^n, z \in \{0,1\}^n. \tag{44}$$

This yields the following statement.

Proposition 3 *A vector $x \in \mathbb{R}^n$ is a weak solution of $\boldsymbol{A}x = \boldsymbol{b}$ if and only if it satisfies (41)-(44) for sufficiently large $\beta \in \mathbb{R}$.*

Proof. Indeed, if x is a weak solution of $\mathbf{A}x = \mathbf{b}$, then it satisfies (41)-(44) for $\beta = \|x\|_\infty$. On the other hand, if x satisfies (41)-(44) for some β, then it satisfies (14) and thus is a weak solution of $\mathbf{A}x = \mathbf{b}$ according to Theorem 3. ■

While the presence of unknown constant β in the above formulation can be taken care of in a numerical implementation of a method based on Proposition 3, this may result in additional computational expenses and numerical errors. On the other hand, we can try to solve the problem of minimizing β subject to the constraints (41)-(44). But then we obtain a mixed integer nonlinear program, which is extremely difficult to solve in general. However, dividing the inequalities in (41)-(43) by 2β (which is positive) and making a change of variables,

$$p = \frac{y}{2\beta},\ \alpha = \frac{1}{2\beta},\ t = \frac{x}{2\beta},$$

we obtain the following equivalent linear mixed 0–1 programming formulation:

$$\max_{\alpha,p,t,z} \alpha \tag{45}$$

subject to:

$$-(\Delta p + \alpha\delta) \le A_c t - \alpha b_c \le \Delta p + \alpha\delta, \tag{45a}$$

$$0 \le p + t \le z, \tag{45b}$$

$$0 \le p - t \le e - z, \tag{45c}$$

$$\alpha \ge 0, \tag{45d}$$

$$\alpha \in \mathbb{R}^1, p, t \in \mathbb{R}^n, z \in \{0,1\}^n. \tag{45e}$$

Define a *minimum norm weak solution* x^* of $\mathbf{A}x = \mathbf{b}$ ($\mathbf{A}x \le \mathbf{b}$) as a weak solution of $\mathbf{A}x = \mathbf{b}$ ($\mathbf{A}x \le \mathbf{b}$) such that $\|x^*\|_\infty \le \|x\|_\infty$ for any other weak solution x of the respective system. Then we obtain the following result.

Proposition 4 *Let $\alpha = \tilde{\alpha}, t = \tilde{t}$ be a part of some feasible solution of (45), and let α^* be the optimal value of α, and t^* be the corresponding vector t.*

(i) If $\alpha^* = 0$, then $\boldsymbol{A}x = \boldsymbol{b}$ is not weakly solvable.

(ii) If $\tilde{\alpha} > 0$ then $x = \tilde{t}/\tilde{\alpha}$ is a weak solution of $\boldsymbol{A}x = \boldsymbol{b}$.

(iii) If $x = 0$ is not a weak solution of $\boldsymbol{A}x = \boldsymbol{b}$ then $\alpha^* < \infty$ and if $\alpha^* > 0$ then $x^* = t^*/\alpha^*$ is a minimum norm weak solution of $\boldsymbol{A}x = \boldsymbol{b}$.

Proof. If $x = 0$ is not a weak solution, then according to (14) there exists index ℓ such that the ℓ-th components of b_c and δ satisfy $|(b_c)_\ell| > \delta_\ell$, and, therefore, from (45a)

$$\alpha \leq \frac{||A||_\infty + ||\Delta||_\infty}{2 \cdot \min\limits_{\ell}\{|(b_c)_\ell| - \delta_\ell \ : \ |(b_c)_\ell| - \delta_\ell > 0\}}.$$

The remainder of the proof follows from Proposition 3 and the discussion following that proposition. ■

The main advantage of this approach over the one in Proposition 3 is that it does not involve any unknown constants. It should be noted that the formulation (45) could also be obtained using the ideas involved in linear mixed 0–1 reformulations for the linear complementarity problem (LCP) [10] and absolute value equations (AVE) [11].

4.2 Checking weak solvability of inequalities

Similar in spirit to the results discussed in the previous subsection, we will use Theorem 5 to formulate an equivalent mixed 0–1 feasibility problem:

$$A_c x - \Delta y \leq \overline{b}, \tag{46}$$

$$0 \leq y + x \leq 2\beta z, \tag{47}$$

$$0 \leq y - x \leq 2\beta(e - z), \tag{48}$$

$$x, y \in \mathbb{R}^n, z \in \{0,1\}^n. \tag{49}$$

Proposition 5 *A vector $x \in \mathbb{R}^n$ is a weak solution of $\boldsymbol{A}x \leq \boldsymbol{b}$ if and only if it satisfies (46)-(49) for some sufficiently large $\beta \in \mathbb{R}$.*

Proof. The proof is analogous to the proof of Proposition 3 and is left as an exercise for the reader. ■

Again, we can formulate a linear mixed 0–1 programming problem,

$$\max_{\alpha,p,t,z} \alpha \tag{50}$$

subject to:

$$A_c t - \Delta p \leq \alpha \overline{b}, \tag{50a}$$
$$0 \leq p + t \leq z, \tag{50b}$$
$$0 \leq p - t \leq e - z, \tag{50c}$$
$$\alpha \geq 0, \tag{50d}$$
$$\alpha \in \mathbb{R}^1, p, t \in \mathbb{R}^n, z \in \{0,1\}^n, \tag{50e}$$

and prove the following statement.

Proposition 6 *Let $\alpha = \tilde{\alpha}, t = \tilde{t}$ be a part of some feasible solution of (50), and let α^* be the optimal value of α, and t^* be the corresponding vector t.*

(i) If $\alpha^* = 0$, then $\boldsymbol{A}x \leq \boldsymbol{b}$ is not weakly solvable.

(ii) If $\tilde{\alpha} > 0$ then $x = \tilde{t}/\tilde{\alpha}$ is a weak solution of $\boldsymbol{A}x \leq \boldsymbol{b}$.

(iii) If $x = 0$ is not a weak solution of $\boldsymbol{A}x \leq \boldsymbol{b}$ then $\alpha^* < \infty$ and if $\alpha^* > 0$ then $x^* = t^*/\alpha^*$ is a minimum norm weak solution of $\boldsymbol{A}x \leq \boldsymbol{b}$.

Proof. Similar to the proof of Proposition 4. ■

5 Control solutions

This section presents two equivalent characterizations of control solutions of a system of linear interval equations. The results are developed similarly to those in the previous section using the characterization from Theorem 7.

Inequality (19) is equivalent to the following mixed 0–1 feasibility problem:

$$-(\Delta y - \delta) \leq A_c x - b_c \leq \Delta y - \delta, \tag{51}$$

$$0 \leq y + x \leq 2\beta z, \tag{52}$$

$$0 \leq y - x \leq 2\beta(e - z), \tag{53}$$

$$x, y \in \mathbb{R}^n, z \in \{0,1\}^n. \tag{54}$$

Proposition 7 *A vector $x \in \mathbb{R}^n$ is a control solution of $\boldsymbol{A}x = \boldsymbol{b}$ if and only if it satisfies (51)-(54) for some sufficiently large $\beta \in \mathbb{R}$.*

Proof. Similar to the proof of Proposition 3. ■

Similarly to the propositions in the above sections, we have:

$$\max_{\alpha,p,t,z} \alpha \tag{55}$$

subject to:

$$-(\Delta p - \alpha\delta) \le A_c t - \alpha b_c \le \Delta p - \alpha\delta, \tag{55a}$$
$$0 \le p + t \le z, \tag{55b}$$
$$0 \le p - t \le e - z, \tag{55c}$$
$$\alpha \ge 0, \tag{55d}$$
$$\alpha \in \mathbb{R}^1, p, t \in \mathbb{R}^n,\ z \in \{0,1\}^n. \tag{55e}$$

Proposition 8 *Let $\alpha = \tilde{\alpha}, t = \tilde{t}$ be a part of some feasible solution of (55), and let α^* be the optimal value of α, and t^* be the corresponding vector t.*

(i) If $\alpha^* = 0$, then $\boldsymbol{A}x = \boldsymbol{b}$ does not have a control solution.

(ii) If $\tilde{\alpha} > 0$ then $x = \tilde{t}/\tilde{\alpha}$ is a control solution of $\boldsymbol{A}x = \boldsymbol{b}$.

(iii) If $x = 0$ is not a control solution of $\boldsymbol{A}x = \boldsymbol{b}$ then $\alpha^* < \infty$ and if $\alpha^* > 0$ then $x^* = t^*/\alpha^*$ is a minimum norm control solution of $\boldsymbol{A}x = \boldsymbol{b}$.

Proof. Similar to the proof of Proposition 4. ■

6 Conclusion

In this paper we formulated several important problems concerned with systems of linear interval equations and inequalities as linear mixed 0-1 programming problems. One key advantage of this approach over the existing methods is that it allows to utilize off-the-shelf mixed integer programming solvers employing state-of-the-art optimization methods. The purpose of this paper was to develop the mathematical foundations for establishing mathematical programming formulations for an important class of problems that have been traditionally approached using other methods. Therefore, we leave the computational comparison of the alternative approaches and, potentially, the development of specialized algorithms for the proposed optimization problems for future work.

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