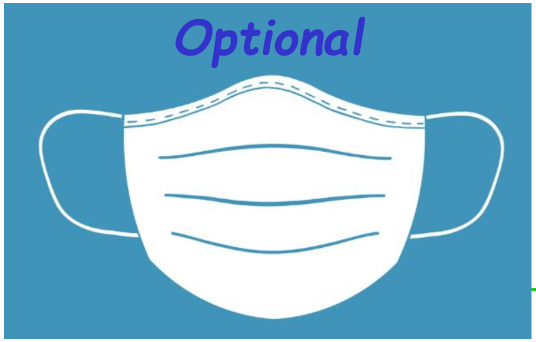
WORCESTER POLYTECHNIC INSTITUTE
MECHANICAL ENGINEERING DEPARTMENT

DESIGN OF MACHINE ELEMENTS
ME-3320, B'2025

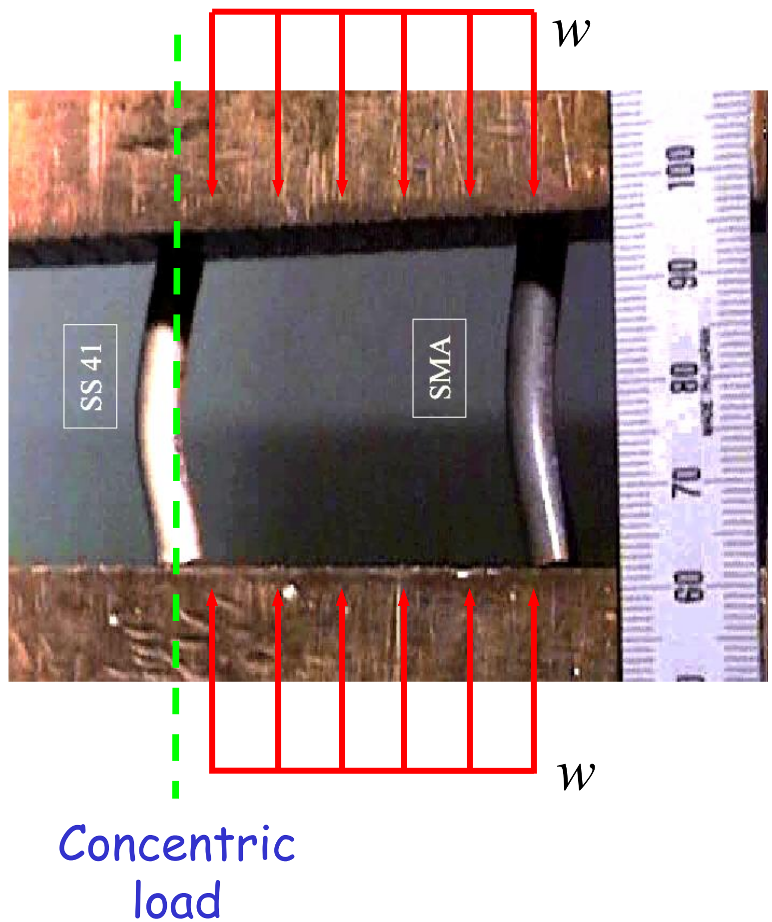
Lecture 08-09

November 2025



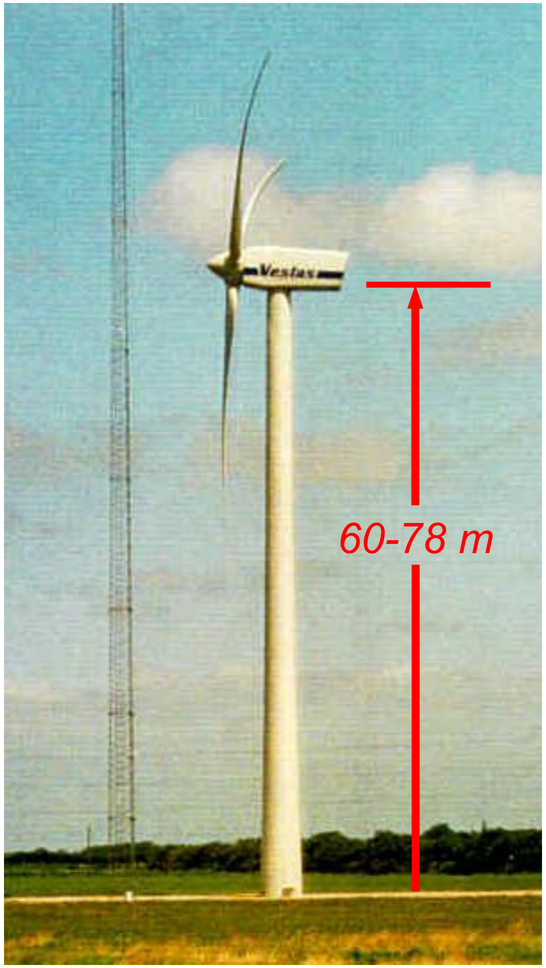
Axial compression -- columns

Examples



Axial compression -- columns
Examples

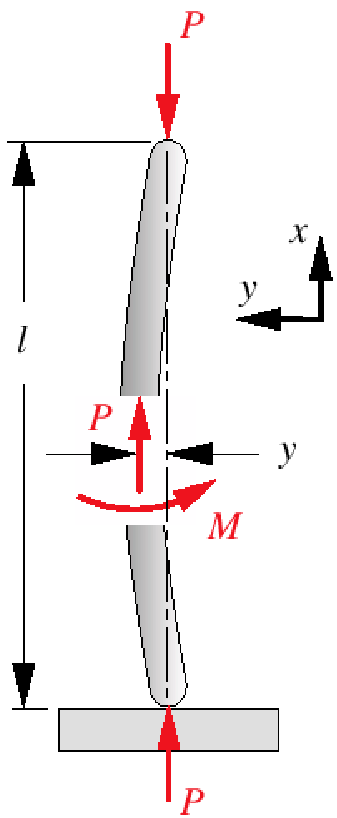
Vestas' V80-2.0 MWatt



Installing a nacelle

Axial compression -- columns
Slenderness ratio: S_r

Buckling of a column



Definitions: $S_r = \frac{l}{k}$, with $k = \sqrt{\frac{I}{A}}$

- *Short columns:* $S_r < 10$

  Calculated based on compression stress criterion: $\sigma_x = \frac{P}{A}$

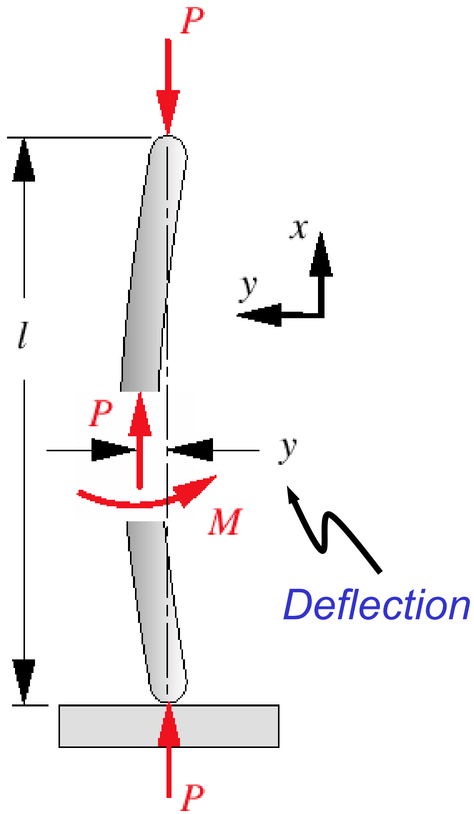
- *Intermediate/long columns:* $S_r \geq 10$

  Calculated based on critical unit load criterion: $\frac{P_{cr}}{A}$

Axial compression -- columns

Long columns: concentric load

Buckling of a column



Bending moment: $M = -P\,y$

For small deflections: $\dfrac{M}{EI} = \dfrac{d^2 y}{dx^2}$

(Governing ODE) ⟹ $\dfrac{d^2 y}{dx^2} + \dfrac{P\,y}{EI} = 0$

Solution (deflection) indicates:

$$y(x) = C_1 \sin\left(\sqrt{\frac{P}{EI}}\,x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}\,x\right)$$

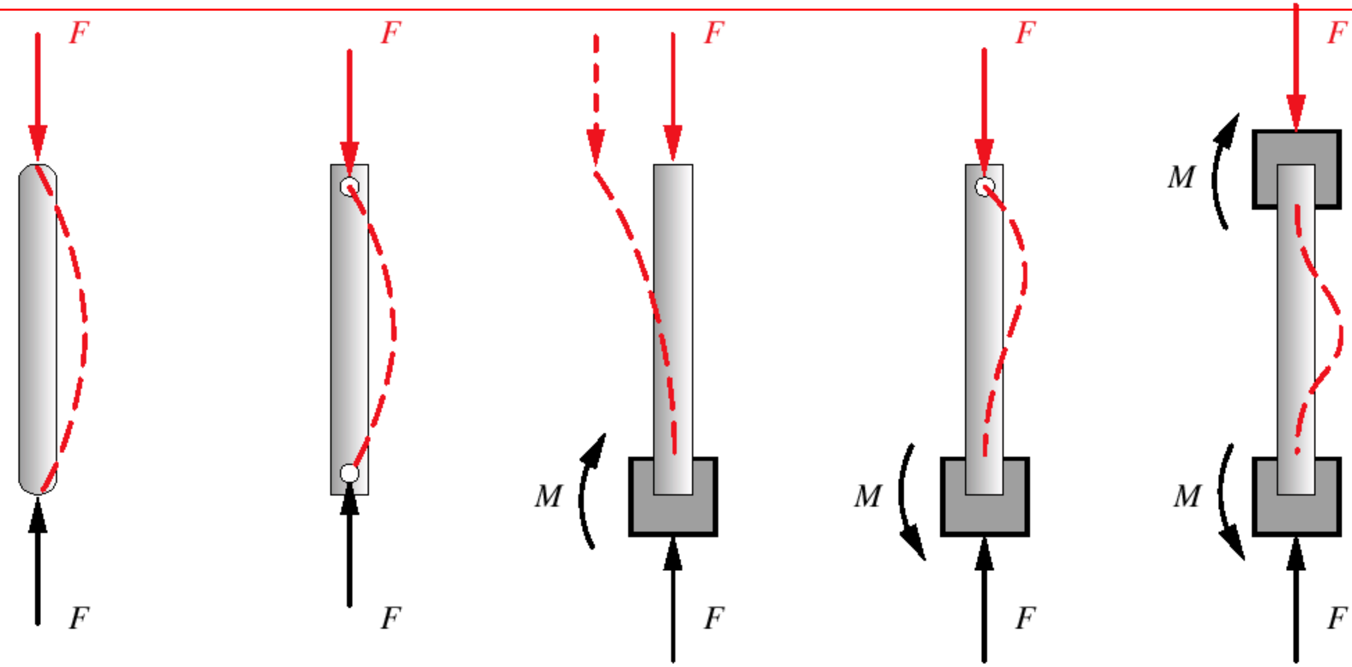
Axial compression -- columns

Long columns

Deflection: $$y(x) = C_1 \sin\left(\sqrt{\frac{P}{EI}}\,x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}\,x\right)$$

C_1 and C_2 are determined from *boundary conditions* (**end conditions**)

Possible end conditions (*make sure you understand BC's in terms of slope and deflection*):



(*a*) Rounded-rounded (*b*) Pinned-pinned (*c*) Fixed-free (*d*) Fixed-pinned (*e*) Fixed-fixed

Axial compression -- columns

Long columns: end conditions + critical load P_{cr}

Deflection: $$y(x) = C_1 \sin\left(\sqrt{\frac{P}{EI}}\,x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}\,x\right)$$

For the rounded-rounded end conditions:

(1) $y(x=0)=0 \longrightarrow C_2 = 0$

(2) $y(x=l)=0 \longrightarrow C_1 \sin\left(\sqrt{\frac{P}{EI}}\,l\right) = 0$

Indicating that:

$$\sqrt{\frac{P}{EI}}\,l = n\cdot\pi; \quad n = 1,2,3\ldots.$$

Many solutions….

Therefore,

$$P_n = \frac{(n\cdot\pi)^2 E\,I}{l^2}; \quad n = 1,2,3\ldots.$$

Many critical loads….

Axial compression -- columns

Long columns: end conditions + critical load P_{cr}

Typically, designs are based on the smallest critical load. Therefore,

$$P_{cr} = \frac{\pi^2 E\, I}{l^2}; \quad \text{for } n = 1$$

using: $I = A\,k^2$ and $S_r = \dfrac{l}{k}$

$$\frac{P_{cr}}{A} = \frac{\pi^2 E}{S_r^2}; \quad \text{for } n = 1$$

Critical ***load per unit area*** *in terms of slenderness ratio*

Corresponding deflection: $y = C_1 \sin\left(\dfrac{\pi\, x}{l}\right)$

For BC's:

(1) $y(x=0)=0$

(2) $y(x=l)=0$

Axial compression -- columns
Accounting for different BC's

In order to take into account other boundary conditions, the concept of ***effective length***, l_{eff} , is introduced:

$$S_r = \frac{l_{eff}}{k} \Longrightarrow \boxed{\frac{P_{cr}}{A} = \frac{\pi^2 E}{S_r^2}}$$

Table 4-3 Column End-Condition Effective Length Factors

End Conditions	Theoretical Value	AISC* Recommended	Conservative Value
Rounded-Rounded	$l_{eff} = l$	$l_{eff} = l$	$l_{eff} = l$
Pinned-Pinned	$l_{eff} = l$	$l_{eff} = l$	$l_{eff} = l$
Fixed-Free	$l_{eff} = 2l$	$l_{eff} = 2.1l$	$l_{eff} = 2.4l$
Fixed-Pinned	$l_{eff} = 0.707l$	$l_{eff} = 0.80l$	$l_{eff} = l$
Fixed-Fixed	$l_{eff} = 0.5l$	$l_{eff} = 0.65l$	$l_{eff} = l$

Designing columns
General procedure

1) Determine force to be supported and expected length of the column (*design objective and corresponding constraint(s)*). Use free-body diagram(s) and equilibrium conditions.

2) Determine cross-section parameters of a proposed column:

- *Area, A*
- *Moment of inertia, I*
- *Radius of gyration, k*

3) Determine slenderness ratio, $S_r = \frac{l_{eff}}{k}$

- *Identify boundary conditions (BC's) and apply appropriate value for the effective length* l_{eff} **--** *use appropriate table (Table 4-3.)*

4) Identify material to use and its corresponding *compressive yield strength*, S_{yc}, and *elastic modulus*, E.

Designing columns
General procedure

4) Determine slenderness ratio at half-yield: $(S_r)_D$ (***Design criterion***)

- $(S_r)_D$ *is obtained as follows:*

  (a) Set the load per unit area as: $$\frac{P_{cr}}{A} = \frac{S_{yc}}{2}$$ *(Half the yield strength value in compression)*

  (b) Therefore, $$\frac{S_{yc}}{2} = \frac{\pi^2 E}{S_r^2}$$

  (c) Solve for slenderness ratio -- using previous equation, (b).
  This is the slenderness ratio at half-yield in compression.

$$(S_r)_D = \pi\sqrt{\frac{2E}{S_{yc}}}$$

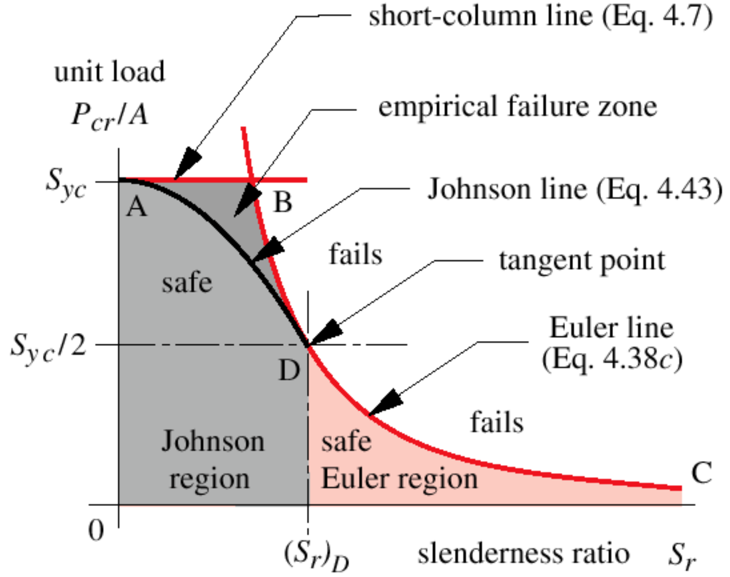
Designing columns
General procedure

5) Determine type of column based on proposed design:

- ***Johnson***: *if* S_r (step 3) < $(S_r)_D$ (Step 4)
- ***Euler***: *if* S_r (step 3) > $(S_r)_D$ (Step 4)

6) Determine critical load

- ***Johnson***: $$P_{cr} = A\left[S_{yc} - \frac{1}{E}\left(\frac{S_{yc}\,S_r}{2\pi}\right)^2\right]$$
- ***Euler***: $$P_{cr} = A\frac{\pi^2 E}{S_r^2}$$



(a) Construction of column failure lines

Designing columns
General procedure

7) Determine allowed load: $$P_{allowed} = \frac{P_{cr}}{SF}$$

- P_{cr} , is the critical load (step 6)
- SF , is the security factor (> 1)

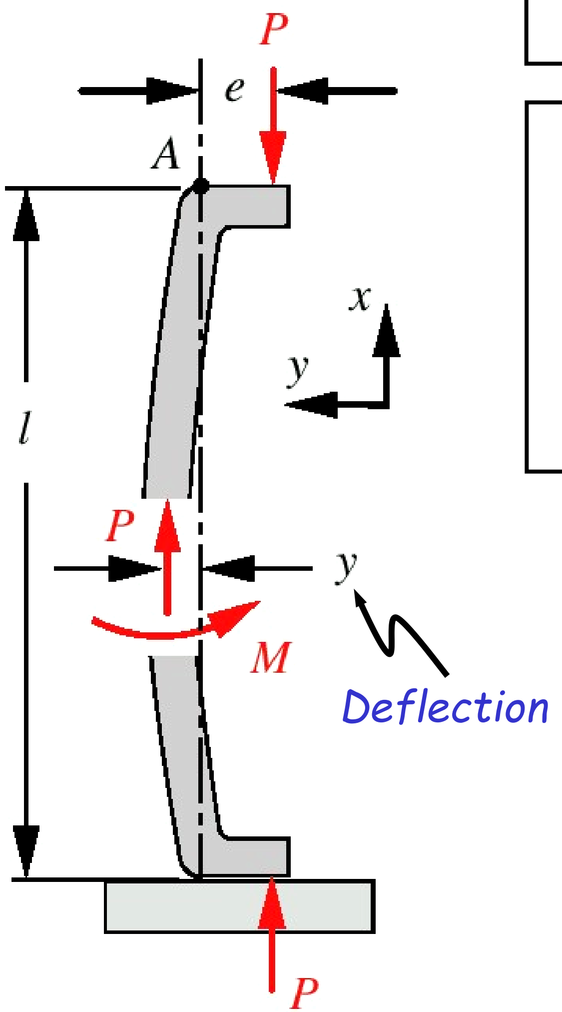
8) If $P_{allowed}$ is > than force to be supported, then, *a satisfactory design has been obtained -- not necessarily the optimal !!* Go to step (1) and refine your design, if possible (e.g., weight minimization)

9) If $P_{allowed}$ is < than force to be supported, then, go to step (1) and improve design. You can select a different section and/or material.

Use of MathCad is strongly recommended !!

Axial compression -- columns
Eccentrically loaded

Eccentrically loaded column



Moments: $\sum M_A = -M + P(e+y) = 0$

For small deflections: $\dfrac{M}{EI} = \dfrac{d^2y}{dx^2}$

(Governing ODE) $\Longrightarrow$ $\dfrac{d^2y}{dx^2} - \dfrac{Py}{EI} = \dfrac{Pe}{EI}$

For BC's: $y(x=0)=0; \quad \left.\dfrac{dy}{dx}\right|_{x=\frac{l}{2}} = 0$

$\Longrightarrow$ $y\left(x=\dfrac{l}{2}\right) = e\left[\sec\left(\dfrac{l}{2}\sqrt{\dfrac{P}{EI}}\right) - 1\right]$

Axial compression -- columns
Eccentrically loaded

Maximum moment: $$M_{\max} = P \cdot \left(e + y\big|_{x=\frac{l}{2}} \right) = P \cdot e \cdot \sec\left(\frac{l}{2} \sqrt{\frac{P}{EI}} \right)$$

Compressive stress: $$\sigma_c = \frac{P}{A} + \frac{M\,c}{I} = \frac{P}{A} + \frac{M\,c}{A\,k^2}$$

with the maximum stress level at $M = M_{\max}$ which yields

$$\sigma_{c,\max} = \frac{P}{A}\left[1 + \left(\frac{e\,c}{k^2} \right) \cdot \sec\left(\frac{l}{k} \sqrt{\frac{P}{4E\,A}} \right) \right]$$ setting $\sigma_{c,\max} = S_{yc}$

$$\longrightarrow \quad \frac{P}{A} = \frac{S_{yc}}{1 + \left(\frac{e\,c}{k^2} \right) \cdot \sec\left(\frac{l_{eff}}{k} \sqrt{\frac{P}{4E\,A}} \right)}$$

Solve for P to obtain the critical load

(Secant column formula)

Reading assignment

- **Chapters 4 of textbook: Sections 4.12 to 4.19**
- **Review notes and text: ES2501, ES2502**

Homework assignment

- **Author's:** as indicated on Website of our course
- **Solve:** as indicated on Website of our course