An Analytical and Numerical Study of Dynamic Materials

A proposal to the
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A. PROJECT SUMMARY

This project focuses on the development and study of dynamic materials, i.e. material composites assembled on a microscale in space and time. Mathematically, the problem is formulated for linear, hyperbolic equations with spatio-temporally varying senior coefficients.

Both analytic and computational means will be applied to the analysis of the effective properties of dynamic materials generated by various microstructures in two or three spatial dimensions and time. Analytically, by applying homogenization, it will be possible to specify attainable bounds for such parameters related to the binary mixtures of isotropic dielectrics in the framework of Maxwell's theory. Computationally, direct numerical simulation of the original equations will be performed to understand the physics of wave propagation through heterogeneous media with complex microstructure. Together, both approaches will lead to the correct formulation and analysis of optimal material design in space-time in response to a dynamic environment.

By allowing spatio-temporal variability in the material constituents, we can create effects that are unachievable through purely spatial design. For example, by appropriately controlling the design factors of a dynamic composite, it is possible to selectively screen large domains in space-time from the invasion of long wave disturbances. One is also able to eliminate the cut-off frequency phenomenon in electromagnetic waveguides; and, dynamic materials may also act towards more effective high frequency power amplification and generation implemented through wave coupling.

Work on dynamic materials will be shared with the larger academic and industrial community via the various activities in which the PI and co-PI are involved. The unique, project-based education of Worcester Polytechnic Institute requires undergraduates, as well as graduate students, to participate in research; and the Center for Industrial Mathematics and Statistics at WPI provides a ready infrastructure through which industrial partners may take part in the development and use of dynamic materials.
C. PROJECT DESCRIPTION

C.1. Introduction to Dynamic Materials

In this proposal, we intend to work in the novel paradigm of spatio-temporal composites or dynamic materials. These are formations assembled from materials which are distributed on a microscale in space and in time. This material concept takes into consideration inertial, elastic, electromagnetic and other material properties that affect the dynamic behavior of various mechanical, electrical and environmental systems. In static or non-smart applications, the design variables, such as material density and stiffness, yield force and other structural parameters are position dependent but invariant in time. When it comes to dynamic applications, we also need temporal variability in the material properties in order to adequately match the changing environment. To this end, in dynamic material design, dynamic materials will take up the role played by ordinary composites in static material design. A dynamic disturbance on a scale much greater than the scale of a spatio-temporal microstructure will perceive this formation as a new material with its own effective properties. With spatio-temporal variability in the material constituents, we shall be able to effectively control the dynamic processes by creating effects that are unachievable through purely spatial (static) material design.

We propose a three-year program of research into the development of general analysis and numerical techniques for the investigation of wave propagation through dynamic materials. Mathematical study makes it possible to predict properties of a composite material even before it is engineered, and consequently, makes it possible to consider developing a composite with optimal properties. Since many details may not be able to be investigated by existing analytical methods, we must also rely on sound numerical techniques to provide approximate solutions to the various equations governing heterogeneous media behaviour. The goal of this project is to develop sound analytical techniques and accurate and reliable numerical algorithms that we can use to describe and, ultimately, regulate the response of dynamic composites to impulse loadings.

Dynamic materials may appear in very diverse physical implementations, including mechanical and electromagnetic. Nevertheless, we may distinguish two principal ways of making them by the spatio-temporal mixing of ordinary materials via the processes of activation and kinetization [1, 17].

Dynamic materials of the first type are obtained by instantaneous or gradual change of the material parameters (stiffness, self-induction, capacitance, etc.) in various parts of the system in the absence of relative motion of those parts. This procedure has been called activation [1, 17] and the corresponding materials termed dynamic materials of the first kind, or activated dynamic materials.

Many examples of activated dynamic materials originate in electrical engineering. As an illustration, consider a transmission line assembled as an array of LC-circuits connected in series with the inductance L and capacitance C changeable in each circuit by switching. The linear inductance and capacitance of the transmission line therefore become changeable in space and in time. More precisely, a pump “wave of linear capacitance” may be generated through the use of p-n junction diodes distributed along the line and appropriately activated in space-time by a relevant commutation scheme [11]. A similar “wave of inductance” may be created through the use of a linear arrangement of non-linear inductances. At this point, mention should be made of nanophase materials, especially magnetic nanoparticles. Such particles exhibit the unique phenomena of superparamagnetism and quantum tunneling of magnetization, accompanied by unusually high coercivities. These effects are observed in a number of magnetic materials such as $\gamma$-Fe$_2$O$_3$ nanocrystals [7] and ferrogels [30] under the room temperature. One can effectively control the inductance of those materials by varying the magnetic field.
Dynamic materials of the second type are obtained when various parts of the system are exposed to relative motion that is prearranged and generated in a certain way. This procedure has been called kinetization, and the relevant materials termed the dynamic materials of the second kind, or kinetic dynamic materials. The materials of this type can be perceived as mixtures of two or more ordinary materials that alternate in space on a microscale; this alternation occurs due to the fact that every constituent participates in its individual material motion. Vibrational motion is most important in this respect; in particular, high frequency standing waves represent a mechanism of creating kinetic dynamic materials. See [2].

There are indeed two-dimensional kinetic materials that are used in practice. We may refer to them as dynamic surfaces. Consider, as an example, a plane formed by two groups of alternating parallel rods. The rods of the first group are attached to one solid body, and the rods of the second group to another. Each group performs a prescribed translational vibration. An extended body laid on such a surface will, under certain conditions, be subject to impact and periodic force with a non-zero average, which is either impossible or very difficult to implement by using a solid vibrating plane. This technique is important toward an effective transportation and screening of granular materials [2].

Generally speaking though, a dynamic material may not be purely activated or kinetic. Instead, its formation may involve both the activation and kinetization processes. Dynamic materials rarely appear to be of natural origin (a living tissue being a notable exception). Probably the main reason for this is that, in order to make the dynamic material work as an element of construction, one should maintain the permanent exchange of energy between the material and its environment. So, the whole system appears to be thermodynamically open. Therefore, dynamic materials should be artificially constructed, and such construction requires special technological solutions. From our ongoing mathematical and numerical study of these novel materials, though, it is clear that they hold a promising means of maintaining control over the properties of the material environment that conduct the dynamical disturbances. As a result, this class of materials will prove invaluable for various engineering and industrial applications.

C.2. Properties and Applications of Dynamic Materials

The properties of dynamic materials may become quite unusual, and as a result their range of applicability is wider than that of pure static formations. In this section, we describe some of the special properties and applications of these novel materials.

I: The Screening Effect Produced by an Activated Elastic Laminate

We can improve the performance of structures by using spatio-temporal composite materials which match the time dependent environment of dynamic problems. For materials experiencing dynamic loads and corresponding surges of stress waves or undesirable impulses, for example, it is important to have a means of screening certain crucial parts of the structure or mechanism from the effects of such surges. It has been shown analytically and numerically in [12, 13] and [27] that, by appropriately controlling the design factors of a dynamic laminate, it is possible to selectively screen large domains in space-time from the invasion of long wave disturbances. With an ordinary static composite, this screening effect is impossible.

Consider an elastic bar assembled from periodically repeated sections of length \( d \), each section composed of two parts each with its individual material properties, density \( \rho \) and stiffness \( k \). We activate this material by bringing the property pattern into motion along the bar at a uniform speed, while keeping the material itself immobile. The moving pattern may be created if additional masses
are attached to and released from the bar at regular intervals, at the frequency which produces
the required \((\rho, k)\) pattern. Wave motion through the bar is governed by the second order wave
equation \((\rho u_t)_t - (ku_x)_x = 0\). Specifically, we make the following assumptions on this composite
medium:
(a) at each point \((z, t)\), the controls \(\rho\) and \(k\) can take either the values \((\rho_1, k_1)\) or \((\rho_2, k_2)\); we refer
to these as ‘material 1’ and ‘material 2’;
(b) these materials are placed within alternating layers in space-time, having the slope \(dz/dt = V\)
on the \((z, t)\)-plane;
(c) the period of the pattern is composed of two successive layers filled, respectively, by materials
1 and 2, the volume fractions of these layers being \(m_1\) and \(m_2\) \((m_1, m_2 \geq 0, m_1 + m_2 = 1)\).
In the \((z, t)\)-plane, the pattern appears as a laminate. Across the layers' interfaces, we assume
continuity of both the displacement of the bar and the density of the momentum flux; both will be
observed if either

\[
V^2 < a_i^2 \quad \text{or} \quad V^2 > a_i^2,
\]

for \(i = 1, 2\), where \(a_i = \sqrt{k_i/\rho_i}\) denotes the speed of waves in material \(i\). These conditions guarantee
that the problem is well-posed, and they give two admissible ranges for \(V\): slow and fast. We are
concerned with the propagation of waves along the activated bar of wavelengths \(\lambda\) that are much
larger than \(d\) \((\lambda \gg d)\).

The relevant analysis may be based either on homogenization, or on the Floquet theory [3, 12,
13]. It shows that low frequency waves perceive the dynamic material as a moving uniform medium
possessing no dispersion; the governing equation appears to be hyperbolic and yields solutions in
the form of d'Alembert waves. Unlike the case of a static laminate \((V = 0)\), the phase velocities of
the d'Alembert waves lack symmetry. When parameters \(k, \rho, m_1, V\) of the pattern fall into certain
ranges, we get \textit{coordinated wave motion}, i.e. the d'Alembert waves both propagate in the same
direction with respect to the laboratory frame. Coordination occurs only if \(\rho_1 \neq \rho_2\) and \(k_1 \neq k_2\).
By switching \(V\) to \(-V\), this common direction will be switched to the opposite. Depending on this
direction, we shall call the relevant pattern the right or left laminate. By applying such laminates,
it becomes possible to selectively screen large portions of the bar from the invasion of low frequency
dynamic disturbances generated either by an initial state, or by a dynamic force applied at the end
of the bar.

We have been successful in developing a numerical method to perform direct numerical simu-
lation of the initial value problem of wave motion through dynamic laminates. The contour plot
of Figure 1 reveals the evolution of a disturbance, initially centered at \(z = 0\), as it splits into two
waves both propagating to the right, thus screening regions to the left of both waves from long
wave disturbances. This picture and the computational details are to be found in [27]. See also

II: Elimination of the Cutoff Frequency in Waveguides and Dielectric Layers

We find that we are able to eliminate the cutoff frequency for waves propagating along waveguides
and across the dielectric layers by activating the dielectric filling. To this end, consider a periodic
array of material layers oriented normally to the \(z\)-axis and filled with isotropic dielectric materials
1 and 2, with material constants \(\epsilon, \mu\), between \(z = \pm t\). The pattern given by such layers is
assumed to move with velocity \(V\) along the \(z\)-axis, so we have an activated spatio-temporal laminate
arrangement. The variations of \(\epsilon, \mu\) in \(z\) and \(t\) are as those described for \(\rho\) and \(k\) in the previous
subsection. The planes \(z = \pm t\) are assumed ideally conducting.
Figure 1: Contour plot of averaged solution to initial value problem of wave propagation through a right laminate: \((k_1, \rho_1) = (1, 1), (k_2, \rho_2) = (10, 9), m_1 = 0.5, V = 0.8\). Wavelength of the disturbance \(\approx 2\), wavelength of material pattern = 0.06

Consider the plane electromagnetic wave with the electric field vector \(\mathbf{E}\) normal to the plane \((i_3, \mathbf{k})\) of incidence where \(\mathbf{k}\) is the wave vector: \(\mathbf{E} = E_2 \mathbf{i}_2\), \(\mathbf{B} = B_1 \mathbf{i}_1 + B_3 \mathbf{i}_3\). When \(V = 0\) (static laminate), waves \(\exp[i(gz + hz + \omega t)]\) much longer than the period of the lamination can propagate along the waveguide only if their frequency \(\omega\) exceeds a certain critical value (the cutoff frequency), so that \(h\) is real. However, for dynamic laminates with \(V \neq 0\), the cutoff phenomenon may be eliminated by a suitable choice of parameters of lamination. The long wave disturbances of any frequency will then travel without damping along the waveguide. A similar effect may be achieved for an activated dielectric layer \(z = \pm \ell\) surrounded by a uniform dielectric with parameters \(\varepsilon_0, \mu_0\); this time, however, the corresponding range of parameters of activation depends on \(\varepsilon_0, \mu_0\) as well.

III: Electromagnetic Wave Coupling

Dynamic materials are capable of not only improving communication through waveguides, but they may also act towards more effective high frequency power amplification and generation implemented through wave coupling. Travelling-wave tubes are widely used to maintain such coupling and energy transport between electronic beams and transmission lines \([11]\). The beam acts in this scheme as a primary circuit in which the energy of an external source (battery) is converted into the energy of moving electrons. This motion generates a slow space-charge wave of negative energy \([21, 22, 25]\) that propagates along the beam. This wave is coupled with the electromagnetic wave in a secondary circuit formed by a transmission line. As a result, the energy of the beam decreases along the tube as it is converted into the electromagnetic energy of a signal in the transmission line. The key factor in this mechanism of conversion is the wave of negative energy maintained in the primary circuit (the beam). However, such a wave may be generated in an activated transmission line by a due choice of activation parameters. Moreover, if we replace the beam by a dynamically activated primary transmission line, then instead of one wave of negative energy in the beam, we may create two such waves in the primary line. Both waves are generated from the energy we pump into this line through activation. Each wave will then participate in coupling with waves in the secondary line, and the effectiveness of coupling will therefore be multiplied.
C.3. Goals

In order to engage in regular work with dynamic materials, we need as much information as possible about their effective material properties. Given a number of original materials that constitute the set $U$, we ask: what will be the set $GU$ of all composites assembled from the elements of $U$, regardless of the microgeometry of the assemblage? Each composite is characterized mathematically by its effective material properties. The set of such properties related to each composite of $GU$ is termed the $G$-closure of $U$ \cite{13, 15} and is also designated as $GU$ with no risk of confusion. If, in addition to this, we demand that each material from $U$ enters the mixture with its own volume fraction $m$, then, instead of $GU$, we arrive at $G_m U$ – the $G_m$-closure of $U$. Finding $G$- or $G_m$-closures, the so-called $G(G_m)$-closure problem, is essential for a complete characterization of the additional material possibilities offered by composites. With regard to spatio-temporal composites, this problem is associated with the linear hyperbolic equations having their senior coefficients variable in space and time. The solution to this problem is a necessary step towards the analysis of more sophisticated problems of mixing that may arise in a wider, particularly non-linear, context.

Once the $G(G_m)$-closure of an original set $U$ is specified, it becomes possible to give a correct formulation of various problems of optimal design, this time in a dynamic environment, for diverse mechanical and electrical systems and devices. In a dynamic environment, we follow the same methodology that has been successfully implemented in a similar elliptic (static) context related to ordinary composites. So far, for the dynamic case, we have a series of results related to one-dimensional wave propagation governed by a standard second order hyperbolic equation of a string type.

Among those results, there is a characterization of the so-called limited $G(G_m)$-closure of an arbitrary set $U$ of isotropic dielectrics in the context of classical electrodynamics. The term “limited” means that not just any two elements of $U$ qualify as participants in material mixing. Some ordering is required to make these elements qualify. These results are discussed in section C.4.3; there are substantial differences from their elliptic counterparts.

(i) To eliminate restrictions mentioned above and thus obtain the full $G(G_m)$-closure of $U$ in the sense of its original definition for one spatial dimension will be the first goal of this project. This extension of $G(G_m)$-closures is important because it reveals one of the key mechanisms of instability in distributed dynamical systems associated with interactions between waves of negative and positive energy (see section C.4.5). The evidence for such instability will be incorporated in the structure of the full $G(G_m)$-closure.

To the best of our knowledge, multi-dimensional spatio-temporal material mixing has never been studied. The relevant analysis will be based upon ideas that have emerged from our study of one-dimensional problems.

(ii) A description of the spatio-temporal $G$-closure, both limited and full, in more than one spatial dimension and time is our second objective. Such a description may be produced, for classical electrodynamics, on the basis of its covariant relativistic formulation cited below in section C.4.1. This formulation applies equally to the analysis of $G$- and $G_m$-closures. For the one-dimensional case, the limited $G_m$-closure of an arbitrary set $U$ appeared to be coincident with the limited $G$-closure of the same set. We want to see if this feature is true also for many spatial dimensions.

(iii) The analysis of the $G_m$-closure problem, both limited and full, in more than one spatial dimension and time is the third objective of this project.

The knowledge of $G(G_m)$-closures allows us to give a correct formulation of some basic problems of optimal material design in dynamics. Such problems may receive a sensible formulation if additional restrictions are imposed on the energy exchange between the system and the external
agent involved in the creation of the dynamic material.

(iv) The correct formulation and analysis of the design problems relevant to spatio-temporal composites is thus the fourth goal of our project.

Along with the mathematical analysis based on homogenization, we use direct numerical solution of the heterogeneous problem, in which we never apply averaging as a computational procedure, to simulate and thus study wave motion through specific, distinct spatio-temporal material configurations. In the one-dimensional context, we have already encountered a number of interesting and challenging numerical issues, such as instability, the intersection of material interfaces, moving grids, etc. We intend to address these issues as we look into individual problems of material design.

(v) The computational analysis of spatio-temporal structures such as a) a semi-infinite elastic dynamic laminate; b) the behaviour at the interface of a left and right laminate; c) a checkerboard microstructure in the \((z,t)\)-plane; d) multiple rank lamination in the \((z,t)\)-plane, will be pursued as the fifth goal of this project. Theoretical homogenization results alone will not give us the details that we need to understand the physics of the problem, so by computationally zooming in on the microscale wave behaviour, we will see how reflection, refraction and transmission work together to produce the unique effects possible with dynamic materials.

(vi) One-dimensional numerical work will be extended to multidimensional dynamic composites. Analytical progress for such composites is possible only for laminates of multiple rank. Numerical computation will be indispensable as a tool for analyzing more general structures.

In the following sections, we give a more detailed description of the future work aimed to pursue the listed goals.

C.4. Analytical Methodology

In this section, we give an illustration of our methodology as applied to classical electrodynamics of moving dielectrics.

As is necessary, we begin with a covariant description of the composites assembled in spacetime. Contrary to the elliptic (static) case where the material tensors are defined in the Euclidean space, in the dynamic (hyperbolic) case, these tensors operate in the Minkowskian space where the fundamental transformations are the Lorentz transformations, the elements of the Lorentz group. These transformations now play the same role that the ordinary rotations, the elements of the Euclidean group, play in the relevant elliptic context. When the material velocities encountered in the hyperbolic case appear to be non-relativistic, the Lorentz group reduces to the Galilean group, with accordingly modified material tensors. In general, the hyperbolic case appears to be conceptually relativistic.

C.4.1. Classification of Spatio-Temporal Composites

In this section, we reproduce a covariant formulation of the Maxwell's theory for moving dielectrics. This scheme will be used in section C.4.3 for a description of the limited \(G\)-closure of an arbitrary set of isotropic dielectrics with regard to one-dimensional wave propagation.

In a standard context \([24]\), the electromagnetic field is defined as two pairs of vectors, \((B, E)\), and \((H, D)\), that generate two skew-symmetric second rank tensors \(F\) and \(f\) in Minkowski's 4-space \(x_1 = z, x_2 = y, x_3 = z, x_4 = ic\) :

\[
F = (cB, -iE) = \sqrt{2}(cB_1a_{12} - cB_2a_{13} - iE_1a_{14} + cB_1a_{23} - iE_2a_{24} - iE_3a_{34}),
\]

\[
f = (H, -icD) = \sqrt{2}(H_1a_{12} - H_2a_{13} - icD_1a_{14} + H_1a_{23} - icD_2a_{24} - icD_3a_{34}).
\]
Here, the symbols \( a_{ik} = (1/\sqrt{2})(e_i e_k - e_k e_i) \), \( i, k = 1, 2, 3, 4 \), constitute an orthonormal basis in the space of skew-symmetric second rank tensors in 4-space constructed from the unit vectors \( e_i, i = 1, \ldots, 4 \), of the Minkowskian frame. The orthonormality means that

\[
a_{ik} : a_{lm} = \begin{cases} 
1, & i = l, k = m, \\
0, & \text{otherwise}. 
\end{cases}
\]

The symbol \( c = 1/\sqrt{\varepsilon_0 \mu_0} \) denotes the velocity of light in vacuum, and \( B_1, \ldots, D_3 \) stand for the relevant vector components along \( e_1, e_2, e_3 \). The Maxwell's system is given by

\[
\frac{\partial F_{ik}}{\partial x_k} = 0, \quad \frac{\partial f_{ik}}{\partial x_k} = 0,
\]

where \( F_{ik} = (1/2)\varepsilon_{iklm} F_{lm} \) is a tensor dual to \( F_{ik} \), with \( \varepsilon_{iklm} \) being a completely antisymmetric fourth rank tensor. We apply here a standard rule of summation over repeated indices.

The tensors \( f \) and \( F \) are linked through the material equation \( f = s : F \), introducing the fourth rank material tensor \( s \). We assume that the material represents an isotropic dielectric with a dielectric permittivity \( \varepsilon \) and magnetic permeability \( \mu \). In a frame immovable with respect to the material, the tensor \( s \) is given by the formula [17]

\[
s = -\frac{1}{\mu c} (a_{12}a_{12} + a_{13}a_{13} + a_{23}a_{23}) - \varepsilon c (a_{14}a_{14} + a_{24}a_{24} + a_{34}a_{34}).
\]  

(2)

When the material with properties \((\varepsilon, \mu)\) is moving at velocity \( V \) relative to the laboratory frame, the tensor \( s \) is given by the formula above, with \( a_{ik} \) replaced by the tensors \( a'_{ik} \) computed for the unit vectors \( e'_1, e'_2, e'_3, e'_4 \) of the moving frame; these vectors are linked with \( e_1, e_2, e_3, e_4 \) through the Lorentz transformation with the velocity \( V \) used as a parameter. This argument incorporates the well-known Minkowski's relations for moving dielectrics.

The eigenvalues, \( \varepsilon c, 1/\mu c \), and the eigentensors, \( a_{ik} \), of \( s \) are assumed to be space and time dependent. Assume that, at each point in space-time, the pair \((\varepsilon, \mu)\) may take only one out of two admissible sets of possible values:

\[
(\varepsilon(x, t), \mu(x, t)) = \begin{cases} 
\varepsilon_1, \mu_1 > 0 & \text{"material 1"}, \\
\varepsilon_2, \mu_2 > 0 & \text{"material 2"},
\end{cases}
\]

(3)

with partitioning of the space-time into subdomains occupied by materials 1 and 2. This partitioning is chosen so as to ensure the existence of solutions \( f, F \) belonging to the relevant Sobolev space, so that we exclude the appearance of shocks in the microstructure. This means that the standard compatibility conditions are observed across the interfaces that separate subdomains in the partitioning. In particular, when a plane wave is travelling along the \( z \)-axis through a pattern of layers that is moving at the velocity \( V \), the Sobolev solutions will exist provided that (1) holds, with \( a_i = 1/\sqrt{\varepsilon_i \mu_i} \) being the velocity of light in material \( i \). This condition certainly restricts the micromechanics of the partitioning but still allows for a broad class of admissible microgeometries.

Formula (2) is used to delineate two types of composites introduced in section C.1. First, there is the case of materials differing only in the eigenvalues \( \varepsilon c, 1/\mu c \) of their \( s \)-tensors; the eigentensors \( a_{ik} \) remain the same for both materials. Since \( a_{ik} \) are preserved, there is no actual material motion. This is the type of partitioning created via activation.

Another type of composite arises when the original materials differ in the eigentensors \( a_{ik} \) alone, with \( \varepsilon, \mu \) being the same for both materials. This difference exists only if the materials are exposed to a relative motion. In the presence of such motion, the eigentensors become different, and we arrive at what may be called a spatio-temporal polycrystal assembled in space-time from the fragments of the same conventional dielectric moving at different speeds. This construction represents an example of kinetization.

C-7
Spatio-temporal polycrystals may be produced due to a substantial anisotropy of every conventional dielectric in space-time; this anisotropy occurs because $\epsilon_c \neq 1/\mu c$ for all materials, the only exception being the vacuum. Both activation and kinetization are combined when both factors - the eigenvalues and eigentensors - are different for the original materials.

C.4.2. Conservation Law for Wave Impedance in One-Dimensional Wave Propagation

Consider one-dimensional wave propagation along the $z_3$-axis where the tensors $F, f,$ and $s$ are defined as $F = \sqrt{2c} (u_{z_3} a_{23} + u_{z_2} a_{24})$, $f = \sqrt{2ci} (v_{z_3} a_{23} - v_{z_2} a_{24})$, and $s = -\frac{1}{\mu c} a_{23} a_{23} - \epsilon c a_{24} a_{24}$ for an immovable material. Here, $u$ and $v$ denote potentials that define the field vectors $E, \ldots, D$ through the formulae $E = i c u_{z_3} e_2$, $B = u_{z_3} e_1$, $H = i c v_{z_3} e_1$, $D = v_{z_2} e_2$. The $s$-tensors for the original materials will differ in their eigenvalues and eigentensors alike; we, however, first consider the pure kinetization case, i.e. the polycrystal defined as an array of fragments of the same original material moving at different speeds along the $z_3$-axis. The velocity pattern of this motion is discontinuous, and the array will also be allowed to move as a whole along the $z_3$-axis at some velocity $V$. The discontinuous velocity pattern may be implemented through the use of the following feasible construction. Consider a linear arrangement of caterpillars placed one after another along the $z$-axis (figure). One half of the track in each caterpillar belongs to the $z$-axis, another does not; the tracks on the $z$-axis become electrically connected, and stay disconnected otherwise. The $z$-axis will then be occupied by material fragments that move each at its own horizontal velocity, and the electric current will flow along the $z$-axis through the assemblage of electrically connected tracks. The performance of the electromagnetic field in a transmission line combined of two such arrangements will be controlled directly by an appropriate velocity pattern. This construction resembles an arrangement of belt transmissions distributed on a microscale along the same direction.

In [17], a conservation law of the wave impedance in one-dimensional spatio-temporal polycrystals was established. Letting $E$ and $M$ denote the effective dielectric permittivity and magnetic permeability of the polycrystal, it has been shown that, regardless of the microstructure in the $(z_3, t)$-space, the effective wave impedance $\sqrt{M/E}$ appears to be the same as the wave impedance $\sqrt{\mu/\epsilon}$ of the original material. This conclusion remains valid in a more general context, when the original materials have different values of $\epsilon$ and $\mu$ (all of them positive), but preserve the same wave impedance $\sqrt{\mu/\epsilon}$. Such materials may form a spatio-temporal composite via activation or kinetization, and the conservation law will nevertheless remain valid. In all such cases, the effective materials constants $E, M$ will obey the inequality $Ec > 1/Mc$, so that $1/E M < c^2$; they will belong to the half of the branch of the hyperbola $E/M = \epsilon/\mu$ in the first quadrant of the $(Ec, 1/Mc)$-plane. This branch spreads from the point $(\sqrt{\epsilon/\mu}, \sqrt{\epsilon/\mu})$ on the diagonal to the point $(\infty, 0)$; certainly, it passes through $(\epsilon c, 1/\mu c)$. All points but one on this branch appear to be attainable, the only exception being the point $(\sqrt{\epsilon/\mu}, \sqrt{\epsilon/\mu})$ on the diagonal. The points with $Ec > \epsilon c, 1/Mc < 1/\mu c$ can be attained by constructing polycrystals of higher rank, with the speed of the material motion not exceeding the velocity of light ($V < 1/\sqrt{\epsilon \mu}$) in each layer. The other part of the branch is also attainable by polycrystals - here $c > V > 1/\sqrt{\epsilon \mu}$. This is the so-called Cherenkov case. The point $(\sqrt{\epsilon/\mu}, \sqrt{\epsilon/\mu})$ cannot be attained by particles with non-zero proper mass, but it may be approached as closely as desired by taking $V$ sufficiently close to $c$. These conclusions equally apply to
polycrystals produced by material with $\epsilon < 0, \mu < 0$. We then arrive at the branch of the hyperbola $E/M = \epsilon/\mu$ that belongs to the third quadrant of the $(E, 1/Mc)$-plane. Original material with positive (negative) values of $\epsilon, \mu$ generates polycrystals lying on the positive (negative) branch of the relevant hyperbola.

The conservation law for the wave impedance can be compared with a similar conservation law known in the analogous elliptic situation. Consider the problem of temperature distribution in a two-dimensional polycrystalline heat conductor produced by a fine scale mixing of the differently oriented fragments of an anisotropic paticular material with the principal heat conductances $d_1, d_2$. The effective heat conductances $\lambda_1, \lambda_2$ of such a plane polycrystal are known to satisfy the conservation law $\lambda_1\lambda_2 = d_1 d_2$ [15].

There is, however, a fundamental difference between these two situations with regard to the attainability issue. In the elliptic case, only the part of the hyperbola $\lambda_1\lambda_2 = d_1 d_2$ that spreads from $(d_1, d_2)$ towards the diagonal is attainable; in particular, an isotropic polycrystal possesses the heat conductance $\sqrt{d_1 d_2}$. This is understandable because a static mixture can only become less anisotropic than the original material. Because of this irreversibility property, only the original material may serve as a paternal substance that generates all mixtures related to the attainable portion of the hyperbola.

The hyperbolic situation is different: as mentioned earlier, any material belonging to the hyperbola, with exception of the point on the diagonal, may serve as a paternal substance generating all other materials on the relevant branch of the hyperbola. This equivalence property, i.e. the absence of a material hierarchy on the hyperbola $E/M = \epsilon/\mu$, allows us to effectively construct what we call the limited $G$-closure of an arbitrary set $U$ of isotropic dielectrics in space-time for one spatial dimension.

**C.4.3. Limited $G(G_m)$-closure of an Arbitrary Set of Original Materials**

We term a material positive (negative), if its material constants $\epsilon, \mu$ are both positive (negative). The limited $G$-closure of a set $U$ of materials is defined as the set of all composites assembled from the elements of $U$ possessing the same sign. A similar definition holds for the limited $G_m$-closure.

As an example, take two materials that are both positive or both negative, with different wave impedances $\sqrt{\mu/\epsilon}$, e.g., $\sqrt{\mu_2/\epsilon_2} > \sqrt{\mu_1/\epsilon_1}$. As shown in [19], by producing activated laminates and polycrystals in space-time, it is possible to generate composites with effective constants $E, M$ occupying the hyperbolic strip

$$\frac{\mu_2}{\epsilon_2} > \frac{M}{E} > \frac{\mu_1}{\epsilon_1},$$

in both the first ($E > 0, M > 0$) and the third ($E < 0, M < 0$) quadrants of the $(E, 1/Mc)$-plane. In other words, if the original materials are both positive or both negative, then it may be possible to produce mixtures of the same or opposite sign depending on the parameters of lamination. The set (4) characterizes the limited $G$-closure of two admissible materials. No material with parameters that do not belong to the set (4) can be assembled in space-time from admissible materials that belong to (4). A dual statement is also true: every material with properties that satisfy (4) can be produced by mixing substances of the same sign from the set.

If we have an arbitrary set $U$ of original materials $(\epsilon_s, \mu_s)$, some positive and some negative, then their limited $G$-closure is entirely specified by the extreme values $\max_s(\mu_s/\epsilon_s)$ and $\min_s(\mu_s/\epsilon_s)$ of this set. Specifically, the limited $G$-closure appears to be the hyperbolic strip $\max_s(\mu_s/\epsilon_s) > \frac{M}{E} > \min_s(\mu_s/\epsilon_s)$.

This condition stands in sharp contrast with the similar result known for the relevant elliptic situation [15]. Consider two anisotropic heat conductors in a plane with principal conductances
\((d_1^{(1)}, d_2^{(1)})\) and \((d_1^{(2)}, d_2^{(2)})\). Their \(G\)-closure in the \((\lambda_1, \lambda_2)\)-plane of effective conductances appears to be a part of the hyperbolic strip \(d_1^{(1)}d_2^{(1)} \leq \lambda_1\lambda_2 \leq d_1^{(2)}d_2^{(2)}\) bounded by the diagonal at one end and by a special curve at another end, this curve representing the rank one laminate made of the original materials. The bound given by this curve is formally generated by the convexity considerations based on the minimal variational principle essential for the elliptic problem. In the hyperbolic case there is no such principle, and the additional bound does not appear. We may, however, introduce the bound artificially if we restrict the freedom of the material motion involved in kinetization by specifying the kinetic energy or another suitable dynamic characteristic of such mixing. This restriction appears to be practically reasonable and may constitute a basis for a sensible formulation of the relevant problems of optimal design. In the elliptic case, we are restricted simply by the fact that the original constituents possess finite values of their material constants.

The limited \(G_m\)-closure of two original materials coincides with the limited \(G\)-closure of the same set. This conclusion, valid for spatio-temporal composites, is substantially different from a relevant result for spatial composites in the elliptic case where the \(G_m\)-closure is known to be only part of the \(G\)-closure [15].

### C.4.4. The Role of Wave Impedance

The analysis carried out for the one dimensional case reveals the fundamental role of the wave impedance as the key factor generating bounds for the limited \(G\)-closure. Physically speaking, this is connected with the phenomenon of reflection: two materials possessing the same wave impedance do not generate reflection of the plane wave travelling in the direction normal to one-dimensional lamellar assemblage of such materials, even if this assemblage is also time-dependent. However, in the multidimensional case with oblique propagation of plane waves, reflection generally occurs, and the wave impedance loses its significance as the key physical factor. At the same time, for a special polarization of the electromagnetic wave, there may be no reflection if the angle of incidence is the Brewster angle. Because this angle is specified by the relative refraction index, we may anticipate that this index will serve as the factor responsible for the \(G\)-closure in the multidimensional case.

### C.4.5. Stability Issues

The creation of dynamic materials may substantially affect stability. The energy flow into the system may either enhance or suppress oscillations, depending on the circumstances. For rapidly changing material properties, the oscillations may grow due to the appearance of waves of negative energy though not necessarily by this mechanism alone. Such waves propagate through a dynamic material with negative material constants, and they are observed in a coordinate frame proper for this material. When coupled to the wave of positive energy in the same coordinate frame, the negative energy decreases, becoming even more negative, whereas the positive energy increases. As a result, the coupling creates instability [22]. This instability will manifest itself through the coupling that arises when we assemble composites from the material constituents of opposite signs in the sense of C.4.3. For this reason, the problem of extending the limited \(G\)-closures to the full \(G\)-closures by allowing for mixing materials of opposite signs, appears to be of special interest and is listed, in Section C.3, as one of our project goals.

### C.5. Numerical Methods

Numerical methods for hyperbolic PDE's with continuous flux functions have been well studied. We use upwind methods that incorporate the physics of the problem into the numerical technique
by exactly or approximately resolving the information carried along the characteristics. It is well understood that a balance must be struck between diffusive first order accurate methods which tend to smear out regions of rapid transition and dispersive second or higher order methods which introduce nonphysical oscillations. For the wave equation with discontinuous flux, there is physical dispersion at the property interfaces due to the partial transmission and reflection of incident waves. So, not only should the schemes that we build adequately address the issues of (nonphysical) numerical diffusion and dispersion, but they must also correctly capture the effects of the physical dispersion. Direct simulation of wave propagation through static heterogeneous media is being studied by LeVeque et al. [6, 10], and in the porous media community [8, 9].

There are special challenges when computing wave motion through spatio-temporal composites, since the flux is discontinuous in time as well as in space, and we make a general note of the following. (i) Numerical and analytical instabilities may occur in situations where the purely spatially heterogeneous problem is stable. (ii) Suitable gridding techniques must be found that will allow for accurate computation of the wave motion and adequate representation of the material structure. (iii) When a material interface intersects with another material interface or a physical boundary, we must take care that the problem remains well-posed and that we enforce correct compatibility conditions. At such regions, we must use special solution techniques.

C.5.1. The General Numerical Technique

We outline an upwind numerical method for computing wave motion through a dynamic material in two spatial dimensions \( x \) and \( z \). The extension (reduction) to three (one) dimensions follows easily. Consider a two dimensional property pattern arranged in a rectangular partitioning, (e.g. a checkerboard configuration), moving with velocity \( \mathbf{V} = \langle V_x, V_y \rangle \). We model the wave motion via the linear two-dimensional wave equation \((\rho u_t)_{t} - (k u_x)_x - (k u_z)_z = 0\). By introducing the auxiliary variables \( v \) and \( w \), we replace the second order equation by the linear first order system

\[
U_t + F(U)_x + G(U)_z = 0,
\]

where

\[
U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad F(U) = \begin{pmatrix} 0 & -1/\rho & 0 \\ -k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad G(U) = \begin{pmatrix} 0 & 0 & -1/\rho \\ 0 & 0 & 0 \\ -k & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix},
\]

and the coefficients of the matrices are discontinuous in space and in time. The compatibility conditions across a moving material interface, \( S(x, z, t) \), are that \( u \) and \( \langle ku_x, ku_z, -\rho u_t \rangle \cdot \nu_S \) are continuous across the interface, where \( \nu_S \) is the space-time normal to \( S \).

The grid cells are rectangular in space and move with the velocity \( \mathbf{V} \), so in space-time the cells are prisms. (Note that for the present problem, we could also make a change of coordinates \((\chi, \zeta) = (x - V_x t, z - V_z t)\) and use a stationary grid.) This Lagrangian grid is indexed by \((i, j)\) and has cell centers located at \((z^n_i, z^n_j)\) at time \( t_n \). The lateral walls of the \((i, j)\)-th space-time grid cell are denoted by \( S_{i+1/2,j} \) and \( S_{i,j+1/2} \) with centers \((z^n_{i+1/2,j}, z^n_{i,j+1/2})\) and \((z^n_{i,j+1/2}, z^n_{i+1/2,j})\) at time \( t_n \). These faces have normals \( \pm \langle 1, 0, -V_x \rangle \) and \( \pm \langle 0, 1, -V_z \rangle \). In the model presented here, the material interfaces coincide with the grid walls so that the material parameters remain constant within each cell. Integrating the first order PDE system over the \((i, j)\)-th space-time prism gives

\[
\Delta z \Delta z U^{n+1}_{i,j} = \Delta z \Delta z U^n_{i,j} - \sum \int \int \langle F(U), G(U), U \rangle \cdot \nu dS,
\]

where the sum is over the four lateral walls of the cell, and \( U^n_{i,j} \) represents the average of \( U \) over the \((i, j)\)-th cell at time \( t_n \). It will be necessary to compute the flux across the cell faces. For example,
for the flux across the face $S_{i+1/2,j}$, the following approximation will be used for
\[
\int_{t_n}^{t_{n+1}} \int_{z_{j-1/2}^n}^{z_{j+1/2}^n} \langle \mathbf{F}(\mathbf{U}), \mathbf{G}(\mathbf{U}) \rangle \cdot \langle 1, 0, -V_z \rangle \, dz \, dt \approx \Delta z \int_0^{\Delta t} (-V_z \mathbf{U} + \mathbf{F}(\mathbf{U})) \, dt.
\]

The argument in the first integrand is $(z_{j+1/2}^n + V_z t, z + V_z t, t)$ and in the second, it is $(z_{j-1/2}^n + V_z t, z, t)$. For a first order method, have the data at time $t_n$ be piecewise constant taking the value $U_{i,j}^n$ on cell $(i,j)$. Thus, the value of $U$ in the integrand is equal to $W(V_z)$ where $U(z,t) = W(z/t)$ is the solution of the Riemann problem $U_t + \mathbf{F}(\mathbf{U})_x = 0$ with initial data and material parameters $U_{i-1,j}$, $(k, \rho)_{i-1,j}$ and $U_{i,j}$, $(k, \rho)_{i,j}$ on the left and the right of the interface.

The Riemann solution will be found via an approximate Riemann solver. We get a field by field decomposition of the $3 \times 3$ system corresponding to material properties on the left and the right of the interface, and we match the solutions by requiring that $u, v$ and $w$ are each continuous across the cell interfaces. It can be readily shown that this numerical continuity condition necessarily gives the analytic compatibility condition. Defining the material impedance as $\sigma = \sqrt{kp}$, the eigenvalue-eigenvector pairs of $\mathbf{F}'(U)$ are $\langle a; (1, -\sigma, 0) \rangle$, $\langle 0; (0, 0, 1) \rangle$, and $\langle -a; (1, \sigma, 0) \rangle$. For a laminate such that $0 < a < V$, for example, the middle state $U$ that is needed in the flux integral will satisfy the following equation once we find the unknowns, $\alpha, \beta, \gamma$:

\[
U = U_{i-1,j} + \alpha \begin{pmatrix} 1 \\ -\sigma_{i-1,j} \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = U_{i,j} - \gamma \begin{pmatrix} 1 \\ \sigma_{i,j} \\ 0 \end{pmatrix}.
\]

A higher order approximation may be obtained by using the piecewise planar reconstruction of the cell information as in [4, 5]. Furthermore, the idea in this technique can be extended to unstructured grids, can be used for the mixing of more than two materials, and can be used for nonlinear flux functions.

**C.5.2. One Dimensional Laminate**

In [27], wave motion through an infinite one-dimensional dynamic laminate with $|V| < a_i$ is modelled. The flux solution found there by tracing characteristics and enforcing continuity of $u$ and its dual variable $v$ across the interface is equivalent to the Riemann solver solution. With the characteristic formulation, though, it can be seen with assurance that the right- and left-going information, $u - v/\sigma$ and $u + v/\sigma$, correctly go through exactly the right amount of reflection and transmission, giving the necessary physical dispersion. We obtain a higher order method by reconstructing $U$ in each cell as linear and using the minmod limiter [26], to control the presence of spurious oscillations. The effective wave velocities computed analytically in [12, 14] are verified and the screening effect is illustrated as in Figure 1. We are also able to observe the long time dispersive wave effects as follows from the analysis of Santosa and Symes for the static problem. The effects of varying $d$, the laminate period, can be seen at http://www.wpi.edu/~sweekees/KL/dynamic.html.

In our ongoing one-dimensional study, we have had to overcome difficulties. In particular, obstacles are encountered in the fast-range ($|V| > a_1, a_2$) problem that are not experienced when in the slow-range case ($|V| \leq a_1$). Additional analysis and further steps had to be taken in order to obtain a stable, reliable numerical algorithm. We anticipate similar and greater problems as we increase the dimensionality of our model.

The case $V = \infty$ is the limiting scenario in the fast-range laminate regime. It describes the situation when the properties in a homogeneous material are switched back and forth between the states $(\rho_1, k_1)$ and $(\rho_2, k_2)$. We call this a temporal laminate [28]. In the fast range and temporal
cases, waves are incident on the material interfaces from one side only, and the physical dispersion acts so that a single incident wave gives birth to two transmitted waves and no reflected ones. These waves are amplified by factors \( T_{2,1} = \frac{c_1 + c_2}{c_1} \) and \( \tilde{T}_{2,1} = \frac{c_1 - c_2}{c_1} \), assuming that the incident wave arises in material 2. The Riemann solution at the interface depends only on data from one side.

In our ongoing work, we have found that, even under the usual CFL constraint for a grid moving with speed \( |V| \), i.e. \( \max_{i=1,2} \frac{(V_i + a_i)\Delta t}{\Delta z} \leq 1 \), we may get instability in our results when \( \sigma_1 \neq \sigma_2 \). This instability is present even in the first order method. Finer scale oscillations which appear in the numerical calculations are quickly magnified in a fashion that is not encountered in static and slow range problems. This may be due to the different type of physical dispersion that occurs in the this class of materials. The early oscillations observed have a period equal to that of the material microstructure \( d \), and for a fixed CFL number, as the mesh size gets smaller, these oscillations make their appearance even earlier. This led us to suspect that this CFL constraint is not sufficient for this class of problem, and a serious stability analysis of numerical methods for linear systems with discontinuous coefficients was necessary.

In [28, 29], we prove via a Fourier/Floquet analysis that a general disturbance imparted to a fast range laminate will exhibit inherent instability, with shorter wave modes growing and longer wave modes remaining stable when \( \sigma_1 \neq \sigma_2 \). Long wave (relative to the laminate period \( d \)) initial disturbances which are stable analytically can be degraded in the computation due to the introduction of shorter wave modes coming from round-off and truncation errors. In the temporal laminate case, it is proven in [28] that the integrity of the numerical results can be retained by using \( \Delta t \) on the order of the laminate frequency and choosing a coarse enough grid (CFL small enough), so that unstable wave modes are not resolved.

Away from the \( |V| = \infty \) limit, however, we are not able to show the existence of a CFL number that prevents the occurrence of oscillations as we did in the temporal case. In [19], a spectral filtering approach is successfully used to compute wave motion through a fast range laminate with negative effective parameters. In coordinates \( \tau = t - \Phi, \zeta = z \), the material is as a temporal laminate where the property pattern depends on \( \tau \) alone, so when a wave is incident on the material interfaces, the two new waves which arise have the same wave number as the incident wave in \( (\tau, \zeta) \) coordinates. The short wave modes which appear are misbegotten and must be destroyed. We perform a spectral decomposition of the initial data and at very regular intervals in the course of the numerical computation, we filter out those wave modes that lie without the range initially present. This spectral approach has proven successful under the usual CFL constraint \( \max_{i=1,2} \frac{(V_i + a_i)\Delta t}{\Delta z} \leq 1 \).

**C.5.3. Some Specific Problems**

In dynamic material problems, the material parameter values are prescribed and controlled by the material designer. As such, we can construct a variety of dynamic composites that give rise to an interesting, wide array of responses to long wave disturbances. Numerical modelling will allow us to investigate, in detail, specific dynamic configurations that give rise to such special effects.

(i) We will consider a **semi-infinite elastic laminate**. At the boundary \( (z = 0) \), we impose the appropriate forms of the standard Dirichlet, Neumann, and free-end boundary conditions. Alternating material layers emanate from or are truncated at the boundary. Some asymptotic analysis on this problem has been performed in [14], but we need to compare these results with direct numerical experiments. We will consider problems such as the following. Assume that for \( z > 0, t > 0 \) the control parameters are set so that we obtain the conditions for a left laminate. One would then expect that there would be no propagation of long wave disturbances into the domain interior. But, what type of motion will we see? *Do we see the effects of high frequency modes?* We
do observe a wave front. What are the characteristics of this front?

(ii) Place a right laminate in the first space-time quadrant \((z > 0, t \geq 0)\), and a left laminate in the second quadrant \((z < 0, t \geq 0)\). We ask, how does a long wave disturbance evolve through the interface of a left and right composite? It is expected that within a sector bounded by the characteristic rays with effective velocities \(-\lambda_l, \lambda_r\) emanating from the origin there would be a shadow zone protected from the invasion of long disturbance. Here \(\lambda_l, \lambda_r\) are the magnitudes of the smaller characteristic speeds of the left/right laminates. What happens to the short and medium wave modes? Will there be long time dispersion effects?

(iii) A rank-\( n \) laminate is a laminate of rank-(\( n-1 \)) structures. The analytical study of rank-2 laminates for hyperbolic equations is a lengthy, bulky calculation and is considerably less flexible and accessible than its elliptic counterpart. When the ranking increases, this problem increases even further in complexity, and we rely on our numerical simulations for a description of effective wave motion through such composites. A rank-\( n \) structure of period \( d \) will contain substructures of order \( d^r \) for \( r = 2, \ldots, n \). Fully direct simulation of wave motion through this material would require us to compute with \( \Delta z = O(d^r) \). This level of detail may make computation quite inefficient when the rank is higher. Instead, we may place a bound on the size of \( \Delta z \) and use an upscaling technique to represent the effect of the fine scale structure up at the \( O(\Delta z) \) level. Upscaling to the cell level requires homogenization techniques.

We have successfully used moving grids to model simple laminates. However, when the material interfaces intersect a solid boundary or other material interfaces, this Lagrangian approach has to be modified. In computations involving boundaries and moving cells, it is inevitable that ‘new’ cells be created by merging and splitting, and we must ensure a fair, conservative distribution of the solution information when we regrid. In the areas containing intersecting regions, we may have to allow a material interface to lie in the interior of the cell rather than being aligned with the cell face. That is, material properties will change within the grid cell. Such cells will have to be marked with the fraction of each material that it contains and how this fraction evolves in time. The solution at time level \( t_n \) may be found by representing the material data in that cell as a uniform averaged value, or by developing an approximate (or exact) solver for the discontinuous problem within the cell. Either of these approaches, if proven sufficiently accurate, can be used for interior cells and therefore, we will be able to use a fixed, Cartesian grid for the entire computational domain.

The numerical methods that we develop in this project can be used for the many problems that contain spatially and/or time varying material parameters. We expect to apply them to problems in areas such as porous media flow, acoustics, waveguides, and even biological tissue studies.

C.6. Integration of Research and Education

The work in this proposal will be shared with the larger academic and industrial community via the various activities in which the PI and co-PI are involved. These activities, together with WPI’s unique, project-based education, will offer extraordinary opportunities for integrating the research with undergraduate and graduate education, for bringing this research to potential industrial users, and for outreach to high school students and teachers.

**Graduate Students** We intend to revise the graduate courses, *Optimal Control and Design with Composite Materials – I and II*, to include dynamic materials as part of its content. We will offer a course on *Numerical Methods for Hyperbolic Differential Equations* with special emphasis on techniques applicable to problems with discontinuous flux. We will advise graduate students who will study both applied mathematics and numerical analysis, and who will then conduct research.
on dynamic materials.

**Undergraduates** Every WPI student must complete an Interactive Qualifying Project (IQP) and a Major Qualifying Project (MQP) under faculty supervision to qualify for graduation. The IQP challenges students to identify, investigate, and report on a topic examining how science or technology interacts with societal structures and values. The MQP is in the student's major area and it should demonstrate the application of the skills, methods, and knowledge of the discipline to solving a problem representative of the type encountered at the professional level. We will work with students on projects related to smart and dynamic materials, and on numerical optimization.

**CIMS and Industry** The PI and co-PI are faculty members of the Center for Industrial Mathematics and Statistics (CIMS) at WPI. Technical experts from companies such as DEKA Research and Development, Compaq, MCI, and United Technologies, come together with CIMS faculty to review the range of problems they face and to develop appropriate research opportunities. As our work and expertise on dynamic materials grows, we will be able to work with the appropriate CIMS industrial partners to develop and promote dynamic materials. See [http://www.wpi.edu/~cims](http://www.wpi.edu/~cims).

**Industrial Math REU** Prof. Weekes is listed as senior personnel in the NSF funded REU Program in Industrial Mathematics and Statistics. This is the only mathematics REU program in the US that exclusively involves students in real-world projects sponsored by corporate partners.

**Strive/Frontiers**

Strive is a two-week, research and learning experience at WPI that challenges high school juniors and seniors to explore the outer limits of knowledge in science, mathematics and engineering. This program is open to students of African American, Hispanic or American Indian descent, and is conducted along with our Frontiers program. Students learn from professors and use WPI's state-of-the-art experimental, analytical and computer technology.

C.7. Results From Prior NSF Support

In 1998, Professor Lurie received NSF award DMS-9803476 in the amount of $90,000 covering the period from 07/01/98 to 06/30/01, for the project entitled "Material Mixing in Space-Time and Dynamic Control in the Coefficients of Linear Hyperbolic Equations." At that time, no data were known about the effective material constants of spatio-temporal composites. As a result of this effort, it has become clear that such constants should be evaluated through the use of a special technique, because the convexity considerations do not work in the hyperbolic case. In one spatial dimension, the fundamental role belongs to the wave impedance as the key parameter that controls the bounds. The wave impedance conservation law established in [17] provided the basis for the construction of the limited $G$-closure of an arbitrary set $U$ of materials and the limited $G_m$-closure of two original materials [20] in one spatial dimension and time. In the course of this analysis, there was established the equivalence property between those materials in the limited $G(G_m)$-closures that have the same sign and possess the same wave impedance. The polycrystals generated by a given material yield all other materials that have the same sign and wave impedance as the original material. The special feature of the limited $G(G_m)$-closures is the absence of bounds other than those specified by the wave impedance; this feature requires that additional constraints be applied to create sensible problems of optimal material design in spacetime. The practical recommendation is to restrict the energy exchange between the system and the external agent generating the spatio-temporal property pattern. These results give impetus for subsequent multidimensional generalizations, including novel physical effects observed in one-dimension. Work arising from this prior NSF support can be found in [1, 13, 14, 16, 17, 18, 19, 20].
References


D-2