Numerical computation of wave propagation in dynamic materials

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Abstract

A higher-order numerical method on a moving grid is used to investigate wave propagation through a one-dimensional dynamic material. Dynamic materials are heterogeneous formations assembled from materials which are distributed on a microscale in space and in time. This concept was introduced in recent research of K. Lurie (1997, 1998, 1999). In particular, we consider wave motion through a material whose property value pattern is a piecewise discontinuous, fast periodic function traveling with constant velocity. In the analytical work of K. Lurie, it was found that by appropriately controlling the design factors of a dynamic composite, it is possible to selectively screen large domains in space–time from the invasion of long wave disturbances. In this paper, we perform direct numerical simulations which validate Lurie’s homogenization results, and present numerical examples to illustrate this screening effect. © 2001 IMACS. Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper, we numerically simulate wave motion through a one-dimensional dynamic elastic composite.

Dynamic materials are composites in which the constituent properties are distributed on a microscale in time, as well as in space. Optimal material design for static or non-smart applications generally results in the formation of composites where the design variables, such as material density, stiffness, yield force, and other structural parameters are position dependent, but invariant in time. The structures that result from these designs are the ordinary composite materials, and their properties depend on the individual properties of the constituent materials and the microgeometry of the mixture. The effective property of a dynamic material, however, also depends on the temporal arrangement, so that by varying the spatio-temporal parameters in the material mixture, we can effect a range of responses.

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The performance of structures can be improved by using spatio-temporal composite materials which match the time dependent environment of dynamic problems. For materials experiencing dynamic loads and corresponding surges of stress waves or undesirable impulses, for example, it is important to have a means of screening certain crucial parts of the structure or mechanism from the effects of such surges. This problem formally reduces to a problem of control of the coefficients in a hyperbolic system of equations. By appropriately controlling the design factors of a dynamic composite, it is actually possible to selectively screen large domains in space–time from the invasion of long wave disturbances [13,16]. With an ordinary static composite, this screening effect is impossible.

One way of making a dynamic material is by implementing either an instantaneous or a gradual change of the material parameters in different parts of the system without any relative movement of those regions. We call this process activation. As an example, consider a transmission line. By mechanical or electronic switching, the line may have its inductance $L$ and capacitance $C$ take one of two admissible pairs of values $(L_1, C_1)$ and $(L_2, C_2)$ at each point of $(x, t)$ space. The variation in capacitance can be effected by the activation of p–n junction diodes which are distributed along the line, and the activation of inductance is accomplished through the use of a ferromagnetic material into which the line is embedded. On the other hand, in the procedure referred to as kinetization, we build a dynamic material by putting parts of the system into actual motion to displace materials in other regions, resulting in a material whose properties are space and time dependent. In most cases, a spatio-temporal composite comes about through a combination of the activation and kinetization processes.

Mathematical treatments are instrumental in understanding the processes that take place in a composite material under applied loading or excitations. The mathematical constructions allow for the prediction of properties of the material even before it is engineered, and consequently, make it possible to consider developing a composite with optimal properties. However, complex and truly novel problems often fall outside the reach of existing analytical methods. We must rely on sound numerical techniques to provide approximate solutions to the various equations governing heterogeneous media behaviour. With efficient and accurate numerical algorithms, we will be able to describe and hence, regulate the response of spatio-temporal composites to impulse loadings.

Our primary purpose in this paper is to present the simulation techniques that we use in our preliminary numerical investigations of spatio-temporal laminates. The numerical study of this new material will (i) help verify the existing homogenization theory done by Lurie, (ii) act as a guide for further analytical study of more complex problems, and (iii) act as an experimental and design tool in the construction of dynamic materials.

The dynamic materials under current study are laminates whose patterns move with a constant velocity $V$. We consider an infinite, dynamic elastic composite with discontinuous density and stiffness coefficients, as described in [13] and in here in Section 2. The governing equation for wave motion through this medium is described by the second-order wave equation

$$\rho \ddot{z} - k \dot{z} = 0,$$

with initial conditions

$$z(x, 0) = f(x), \quad z_x(x, 0) = g(x).$$

The density $\rho$ and stiffness $k$ are fast periodic functions given by

$$\rho(x, t) = \rho \left( \frac{x - V t}{\epsilon} \right), \quad k(x, t) = k \left( \frac{x - V t}{\epsilon} \right),$$

(1)
where $\epsilon$ is a small parameter. The property pattern moves with uniform constant velocity $V$ and $\rho(\xi)$ and $k(\xi)$ are 1-periodic functions. At $t = 0$, the density and stiffness have a periodic pattern, and (2) may be perceived as this pattern being activated, i.e., brought into motion with velocity $V$ along the $x$-axis. Wave propagation through the static laminate is a special case ($V = 0$) of the problem above. This static problem has been analytically studied in [23,9,20,22,23].

Numerical methods for hyperbolic PDE’s with continuous flux functions have been well studied [7,10,25]. Upwind methods incorporate the physics of the problem into the numerical technique by exactly or approximately resolving the information carried along the characteristics. First-order accurate methods introduce numerical diffusion into their solutions, so profiles are smeared and there is a loss in the resolution of important features like discontinuities. Second-order schemes contain much less numerical diffusion, so discontinuities in the solution are better resolved. However, these schemes have a dispersive property which causes spurious oscillations in regions of rapid transition. We can build high resolution methods that are at least second order accurate in smooth solution regions, while giving non-oscillatory sharp transition regions. This is done by modifying the purely second order method by adding controlled or ‘limited’ contributions from a first order technique.

For the wave equation with discontinuous, but linear flux, we must build schemes that adequately address the issues of numerical diffusion and dispersion. In addition, there is physical dispersion at the property interfaces resulting in the partial transmission and reflection of incident waves, so the schemes must be able to properly handle these effects as well.

Direct simulation of the linearized acoustic problem with drastically varying material parameters was studied by Fogarty and LeVeque in [5,6]. In the case of periodic media, this is essentially the problem given by (1) and (2) when $V = 0$. They use a high-resolution finite-volume method based on the fluctuation splitting approach presented by LeVeque in [11] which has been implemented in the CLAWPACK software package [12]. However, the standard limited methods in CLAWPACK yielded numerical instabilities for this problem. A new method of limiting called transmission-based limiting was introduced as a fix.

In this work, we shall not follow the fluctuation splitting approach of LeVeque. Since the problem is linear, it is relatively straightforward to focus on the propagation of the characteristic variables. The second order version of the scheme relies on reconstructing linear profiles of the characteristic variables in each computational cell, and limiting these slopes via the minmod limiter. With the minmod slope limiting method, we have not encountered the numerical instability referred to in [5,6], and have not had to devise a new type of limiting. However, further work needs to be done in order to understand the behaviour of limiters in the case of discontinuous flux coefficients.

To simulate wave motion through a dynamic ($V \neq 0$) laminate, we employ a grid moving with the property pattern speed $V$. The material properties change only across cell interfaces. A higher-order scheme is even more necessary on moving grid advection problems, since the grid velocity contributes to the numerical diffusion of the schemes.

In the following section, we describe more precisely the problem that we are simulating. The details of the numerical method are given in Section 3. In Section 4, we give the results of some numerical experiments which illustrate the use and potential of the method for studying the properties of dynamic materials, and provide some concluding remarks in Section 5.
2. Problem formulation

In this section, we present the mathematical formulation of the problem of wave propagation through spatio-temporal laminates. We consider the wave equation (1)

$$\rho \frac{\partial^2 z}{\partial t^2} - k \frac{\partial^2 z}{\partial x^2} = 0,$$

and make the following specific assumptions about the composite medium:

(a) at each point \((x, t)\), the controls \(\rho\) and \(k\) can take either the values \((\rho_1, k_1)\) or \((\rho_2, k_2)\); we refer to these as 'material 1' and 'material 2';

(b) these materials are placed within alternating layers having the slope \(\frac{dx}{dt} = V\) on the \((x, t)\)-plane;

(c) the period of the pattern is composed of two successive layers filled, respectively, by materials 1 and 2, the volume fractions of these layers being \(\alpha_1\) and \(\alpha_2\).

See Fig. 1. The slope \(V\) is chosen so as to ensure the regular transition of continuous disturbance \(z(x, t)\) through the interface from one layer to another. For smooth solutions, kinematic and dynamic compatibility conditions of continuity across the material interfaces are enforced. These conditions are satisfied if the following relationship between the characteristic speeds of the materials \(a_i = \sqrt{k_i/\rho_i}\), and the material pattern speed \(V\) holds [13]:

$$\text{sgn}(a_1 - |V|) = \text{sgn}(a_2 - |V|).$$

We are thus able to work in two regimes—a subsonic regime where \(|V| < a_1, a_2\) and a supersonic regime where \(|V| > a_1, a_2\). We limit the presentation in this paper to the subsonic regime.

In [13], Lurie uses a standard analytical homogenization procedure [1,18] to calculate the effective phase velocities and the governing differential equations of such a composite when the period of the medium, \(\varepsilon\), is much smaller than the wavelength of the initial disturbance. Using the notation \(\langle \xi \rangle = m_1 \xi_1 + m_2 \xi_2\) and \(\bar{\xi} = m_1 \xi_2 + m_2 \xi_1\), we have the following differential equation for \(\langle z \rangle\), the value of the disturbance \(z\) averaged over the period of the array:

$$\frac{1}{a_1 a_2} \left[ V^2 - k \left( \frac{T}{\rho} \right) \right] \langle z \rangle_{tt} + 2V \left[ \frac{T}{\rho} \left( \frac{T}{k} \right) - \left( \frac{T}{a^2} \right) \right] \langle z \rangle_{tx} - \rho \left( \frac{T}{k} \right) \left[ V^2 - \frac{1}{\bar{\rho}(1/k)} \right] \langle z \rangle_{xx} = 0. \quad (4)$$
When inequality (3) holds, the equation above is hyperbolic. Like its uniform material counterpart, this dynamic material will allow for D’Alembert wave solutions traveling with characteristic phase velocities. These velocities \( \lambda_1, \lambda_2 \) are calculated from the p.d.e. above. We note that Floquet analysis is also applicable to spatio-temporal composites and, in [17], Lurie uses the theory to recompute the wave velocities:

\[
\lambda_{1,2} = \frac{V a_1^2 a_2^2 [\sigma (1/k) - (1/a^2)] \pm a_1 a_2 \sqrt{\sigma (1/k) (V^2 - k/\rho) (V^2 - (1/\rho)/(1/k))}}{V^2 - k(1/\rho)}. \tag{5}
\]

It is possible to choose the problem parameters \( \rho_1, \rho_2, k_1, k_2, m_1, V \) so that \( \text{sgn} (\lambda_1) = \text{sgn} (\lambda_2) \). The D’Alembert waves which arise will thus travel in the same direction relative to a laboratory frame. This coordinated wave motion emerges if either

\[
\frac{1}{\sigma (1/k)} < V^2 < a_1^2 \quad \text{or} \quad a_2^2 < V^2 < k \left( \frac{1}{\rho} \right),
\]

where it is assumed that \( a_1 < a_2 \). With the aid of this coordinated wave motion, it becomes possible to protect some extended domains in space–time from the invasion of long wave dynamic disturbances. The general disturbance will then become filtered according to its frequency spectrum, and the higher harmonics alone will be allowed into certain regions in space–time.

In the case \( V = 0 \), the effective medium equation for the wave problem (1) is

\[
\langle \rho \rangle_{tt} - (1/k)^{-1} \langle z \rangle_{xx} = 0,
\]

yielding effective wave motion with velocities \( \pm \Omega \), where

\[
\Omega^2 = (1/k)^{-1} \langle \rho \rangle^{-1}.
\]

This is a well-known result and is a specific case of (5). Note that the screening effect is impossible for the static problem since the effective wave velocities are of opposite signs.

Eq. (1) is equivalent to the system

\[
z_t - \frac{1}{\rho} v_x = 0, \tag{6}
\]

\[
v_t - k z_x = 0, \tag{7}
\]

which can be decoupled into two advection equations in the characteristic variables \( z - v/\sqrt{k\rho} \) and \( z + v/\sqrt{k\rho} \):

\[
\begin{pmatrix}
  z + v/\sqrt{k\rho} \\
  z - v/\sqrt{k\rho}
\end{pmatrix}
+ \begin{pmatrix}
  -\sqrt{k/\rho} & 0 \\
  0 & \sqrt{k/\rho}
\end{pmatrix}
\begin{pmatrix}
  z + v/\sqrt{k\rho} \\
  z - v/\sqrt{k\rho}
\end{pmatrix} = \begin{pmatrix} 0 \\
  0 \end{pmatrix}. \tag{8}
\]

The information \( z + v/\sqrt{k\rho} \) travels along characteristics with velocity \( -\sqrt{k/\rho} \), whereas \( z - v/\sqrt{k\rho} \) travels along characteristics at velocity \( \sqrt{k/\rho} \). We will use \( \mu \) to denote the material ‘impedances’, \( \sqrt{k\rho} \). At any point \( (\tilde{x}, \tilde{t}) \), the values of \( z \) and \( v \) (denoted by \( \tilde{z}, \tilde{v} \), are completely determined by the characteristic information arriving there:

\[
\tilde{z} - \tilde{v}/\mu_L = z_L - v_L/\mu_L, \tag{9}
\]

\[
\tilde{z} + \tilde{v}/\mu_R = z_R + v_R/\mu_R. \tag{10}
\]

Here, \( z_L - v_L/\mu_L \) is the value of \( z - v/\sqrt{k\rho} \) that is constant along the characteristic of speed \( a_L \) which originates to the left of \( \tilde{x} \) and which reaches \( (\tilde{x}, \tilde{t}) \) through pure material of constant type \( (k_L, \rho_L) \).
Similarly, \( z_R + v_R/\mu_R \) is the value of \( z + v/\sqrt{k\rho} \) that is constant along the characteristic of speed \( \alpha_R \) coming from the right of \( x \) through pure material of constant type \((k_R, \rho_R)\). If \((x, t)\) is in the interior of material \( i \) region \((i = 1 \text{ or } 2)\), then \((k_i, \rho_i) = (k_R, \rho_R) = (k, \rho)\). Given the kinematic and dynamic compatibility conditions that \( z \) and \( v \) are continuous along the material interfaces, on an interface with material 1 on the left and material 2 on the right, say, one has \((k_{L}, \rho_{L}) = (k_{1}, \rho_{1}) \text{ and } (k_{R}, \rho_{R}) = (k_{2}, \rho_{2})\).

The numerical technique presented in this paper relies squarely on the understanding of the propagation of the characteristic data.

3. Numerical method

In this section, we describe the numerical method that we have designed to approximate the solutions \( z(x, t) \) to (1), (2) on an infinite domain for \( t > 0 \). The coefficients \( \rho \) and \( k \) are piecewise constant, fast periodic functions as described in the previous section. The method is a conservative, finite volume, upwind method on a moving grid.

3.1. Grid

We employ a moving one-dimensional grid. The grid points lie in the centers of grid cells of width \( \Delta x_i \) and \( \Delta x_{j+1/2} \) such that a layer of material \( i \) has a given number of cells of width \( \Delta x_i \) for \( i = 1, 2 \). The grid center at time \( t_n = n\Delta t \) of cell \( j \) is denoted by \( x^n_j \). The width of the \( j \)th grid cell is \( \Delta x_j \) and this value depends on whether this cell is in material 1 \((\Delta x_j = \Delta x_1)\) or in material 2 \((\Delta x_j = \Delta x_2)\). At the initial time \( t_0 = 0 \), the grid is constructed so that the material properties change across the cell interfaces only. They do not change values in the interior of a cell. The grid points move with the pattern velocity \( V \) such that \( x^n_j = x^0_j + V t_n \). The values of \( z \) and \( v \) at the grid points, denoted by \( z^j_n, v^j_n \), simultaneously represent the approximations to \( z \) and \( v \) at those points, as well as the values of \( z \) and \( v \) averaged over cell \( j \).

The generic cell volume in space–time is a parallelogram with lateral sides having slope \( \text{d}x/\text{d}t = V \), and base and top at time levels \( t_n \) and \( t_{n+1} \). Let

\[
I_{j+1/2}(t) = x^n_{j+1/2} + V(t - t_n) = x^0_{j+1/2} + V(t - t_0),
\]

denote the right lateral side of the \( j \)th space–time cell volume, where \( x^n_{j+1/2} = x^n_j + \frac{1}{2}\Delta x_j = x^n_{j+1} - \frac{1}{2}\Delta x_{j+1} \). A material interface will lie along an \( I_{j+1/2} \).

3.2. Evolution

To obtain the cell averaged values of \( z \) and \( v \) at time \( t_{n+1} \) in cell \( j \), we integrate the conservation law (6), (7) (or equivalently, (8)) over the space–time parallelogram volume associated with the \( j \)th cell. This yields

\[
z^{n+1}_j = z^j = \frac{1}{\Delta x_j \rho_j} \int_{t_n}^{t_{n+1}} \left[ v(I_{j+1/2}(t), t) - v(I_{j-1/2}(t), t) \right] \text{d}t.
\]
\[
\frac{1}{\Delta x_j} V \int_{t_n}^{t_{n+1}} \left[ z(I_{j+1/2}(t), t) - z(I_{j-1/2}(t), t) \right] dt, \tag{12}
\]

\[
v_j^{n+1} = v_j^n + \frac{1}{\Delta x_j} k_j \int_{t_n}^{t_{n+1}} \left[ z(I_{j+1/2}(t), t) - z(I_{j-1/2}(t), t) \right] dt \]

\[
+ \frac{1}{\Delta x_j} V \int_{t_n}^{t_{n+1}} \left[ v(I_{j+1/2}(t), t) - v(I_{j-1/2}(t), t) \right] dt. \tag{13}
\]

The flux expressions

\[
F(U)(I_{j+1/2}(t), t) - V U(I_{j+1/2}(t), t),
\]

where \( U = (z, v) \) and \( F(U) = (v/\rho, k z) \), bring to mind the important paper of Osher [19] in which he derives a closed form expression for the solution to the scalar Riemann problem. See [19, Lemma 1.1], and also the paper of Sanders [21].

Conservative techniques distinguish themselves from each other by their approximations to the flux integral. Since our equations (8) are linear, we calculate the values of \( z, v \) along the cell interfaces in a straightforward manner by tracing characteristics, i.e., by solving system (9)–(10). On interface \( I_{j+1/2} \), we trace back to the left within cell \( j \) and to the right in cell \( j + 1 \). For the characteristic information from the \( j \)th and \((j + 1)\)th grid cell at time \( t_n \) to reach interface \( I_{j+1/2} \) between times \( t_n \) and \( t_{n+1} \), we must have

\[
I_{j+1/2}(t_{n+1}) - a_j \Delta t \in [x_{j-1/2}^n, x_{j+1/2}^n],
\]

\[
I_{j+1/2}(t_{n+1}) + a_{j+1} \Delta t \in [x_{j+1/2}^n, x_{j+3/2}^n],
\]

where \( a_j, a_{j+1} \) are the characteristic speeds in cells \( j \) and \( j + 1 \). Considering (11) this gives the CFL restriction

\[
\max_{i=1,2} \frac{(|V| + a_i) \Delta t}{\Delta x_i} \leq 1,
\]

allowing us to compute \( z \) and \( v \) at \( I_{j+1/2}(t), t \) for \( t \in [t_n, t_{n+1}] \):

\[
(z - v/\mu_j)(I_{j+1/2}(t), t) = (z - v/\mu_j)(I_{j+1/2}(t) - a_j(t - t_n), t_n), \tag{16}
\]

\[
(z + v/\mu_{j+1})(I_{j+1/2}(t), t) = (z + v/\mu_{j+1})(I_{j+1/2}(t) + a_{j+1}(t - t_n), t_n). \tag{17}
\]

The first-order method comes about by reconstructing the profile to the approximate solution of \( z(x, t_n) \) and \( v(x, t_n) \) as piecewise constant. In cell \( j \), \( z \) and \( v \) take on the cell-averaged values of \( z_j^n \) and \( v_j^n \). For \( t \in [t_n, t_{n+1}] \) then, the values of \( z \) and \( v \) along interface \( I_{j+1/2} \) depend on the quantities \( z_{j+1}^n - v_{j+1}^n/\mu_j \) and \( z_{j+1}^n - v_{j+1}^n/\mu_{j+1} \). We denote the interface values as \( z_{j+1/2}, v_{j+1/2} \), and compute them by solving system (16), (17):

\[
\begin{pmatrix}
1 \\
\frac{1}{\mu_j}
\end{pmatrix}
\begin{pmatrix}
z_{j+1/2}
\end{pmatrix}
= \begin{pmatrix}
z_j^n - \frac{v_j^n}{\mu_j}
\end{pmatrix},
\]

\[
\begin{pmatrix}
1 \\
\frac{1}{\mu_{j+1}}
\end{pmatrix}
\begin{pmatrix}
v_{j+1/2}
\end{pmatrix}
= \begin{pmatrix}
z_{j+1}^n + \frac{v_{j+1}^n}{\mu_{j+1}}
\end{pmatrix}. \tag{18}
\]
The integrals of $z$ and $v$ along the interfaces in Eqs. (12), (13) are therefore exactly $\Delta t z_{j+1/2}$ and $\Delta t v_{j+1/2}$, when $z$ and $v$ are constant in each cell. The first order method is, therefore,

$$z_j^{n+1} = z_j^n + \frac{\Delta t}{\Delta x_j} \left[ v_{j+1/2} - v_{j-1/2} \right] + \frac{\Delta t}{\Delta x_j} V [z_{j+1/2} - z_{j-1/2}],$$

$$v_j^{n+1} = v_j^n + \frac{\Delta t}{\Delta x_j} k_j [z_{j+1/2} - z_{j-1/2}] + \frac{\Delta t}{\Delta x_j} V [v_{j+1/2} - v_{j-1/2}].$$

(19)

(20)

It is easy to check that system (18) is equivalent to

$$\begin{pmatrix} 1 & 1 - \frac{1}{\mu_{j+1}} \\ \frac{1}{\mu_j} & 0 \end{pmatrix} \begin{pmatrix} z_j^{n+1} \\ v_j^{n+1} \end{pmatrix} = \begin{pmatrix} R_{j+1,j} \\ T_{j+1,j} \end{pmatrix} \begin{pmatrix} z_j^n - \frac{v_j}{\mu_j} \\ v_j^{n+1} - \frac{z_j^n}{\mu_{j+1}} \end{pmatrix}.$$  

(21)

Here $R_{a,b}, T_{a,b}$, defined as

$$R_{a,b} = \frac{\mu_a - \mu_b}{\mu_a + \mu_b}, \quad T_{a,b} = \frac{2\mu_a}{\mu_a + \mu_b},$$

(22)

are the reflection and transmission coefficients for a wave moving from material $a$ to material $b$ across an interface that is itself moving with speed $|V| < a_1, a_2$. Thus, resolving the characteristic information arriving at the material interfaces as we do in (16), (17) or (18) is equivalent to formulating our compatibility conditions in terms of transmission and reflection of right-going and left-going waves:

$$z_{j+1/2} - \frac{v_{j+1/2}}{\mu_{j+1}} = T_{j+1,j} \left( z_j - \frac{v_j}{\mu_j} \right) + R_{j+1,j} \left( z_{j+1} + \frac{v_{j+1}}{\mu_{j+1}} \right),$$

$$z_{j+1/2} + \frac{v_{j+1/2}}{\mu_j} = R_{j+1,j} \left( z_j - \frac{v_j}{\mu_j} \right) + T_{j+1,j} \left( z_{j+1} + \frac{v_{j+1}}{\mu_{j+1}} \right).$$

That is, one would require that the characteristic information $(z - v/\mu_{j+1})$, originating at the interface, which moves to the right into cell $j + 1$ should be composed of the transmitted portion, $T_{j+1,j}$, of $(z - v/\mu_j)$ coming from cell $j$ which has made its way into cell $j + 1$, plus the portion, $R_{j+1,j}$, of the originally left-going information $(z + v/\mu_{j+1})$ in cell $j + 1$ that has been reflected back into that cell. Similarly, the information $(z + v/\mu_j)$ originating at the interface which moves left into cell $j$ should be composed of the transmitted portion, $T_{j+1,j}$, of $(z + v/\mu_{j+1})$ coming from cell $j + 1$ which has made its way into cell $j$, plus the portion, $R_{j+1,j}$, of the originally right-going information $(z - v/\mu_j)$ in cell $j$ that has been reflected back into that cell.

The calculation of reflection and transmission coefficients for the subsonic problem $|V| < a_1, a_2$ is carried out in the appendix to this paper. Fig. 2 illustrates the separation of an incident wave into reflected and transmitted portions. We remark that while the coefficient values $T_{a,b}$ and $R_{a,b}$ are the same in the stationary problem (see [4]) as they are in the subsonic problem, they are different in the supersonic case $|V| > a_1, a_2$.

To see in detail the features of the problem solution without using a very fine mesh, one must use a scheme that provides more numerical accuracy than a first order method. Numerical diffusion smears out the solution profile and it becomes difficult to distinguish between the natural, physical effects produced by this new dynamic material from the numerical ones. For example, the numerical solution to the initial value problem when $\varepsilon$ is much smaller than the wavelength of the initial wave disturbance shows quite
clearly the D'Alembert waves which arise as predicted by Lurie. However, the amplitude of these two waves do not appear to be equal in the dynamic problem, as they are in the static version. Is the unequal amplitude a physical result or a numerical artifact?

If we derive the modified equations for the first-order upwind method for the scalar advection problems $u_t + au_x = 0$ and $u_t - au_x = 0$ on a grid moving with velocity $V$:

$$u_t + au_x = \frac{\Delta x}{2}(a - V) \left[ 1 - \frac{\Delta t}{\Delta x}(a + V) \right] u_{xx} + O(\Delta x^2),$$

$$u_t - au_x = \frac{\Delta x}{2}(a + V) \left[ 1 - \frac{\Delta t}{\Delta x}(a - V) \right] u_{xx} + O(\Delta x^2),$$

we see that for a given $V \neq 0$, $\Delta t$, and $\Delta x$, the numerical diffusion is unequal for the two problems. So, before using the numerical results to make inferences about the wave amplitudes for our dynamic second-order problem, one would have to take care to note that the characteristic data propagating at opposite speeds, $\pm a_i$, are experiencing different levels of diffusion due, in part, to the use of a moving grid.

Thus, to be able to more efficiently and accurately use numerical experiments to study dynamic materials, it is essential to build higher-order schemes.

3.3. Higher-order method

Reconstructing the discrete pieces of characteristic data, $(z \pm v/\mu_j)^n$, as linear over each cell, leads to a second-order scheme. At each time level $t_n$, we choose slopes $\sigma_{j,\pm}^n$ of the profiles of $(z \pm v/\mu_j)^n = Q_{j,\pm}^n$ in cell $j$ so that the data is seen as

$$z - v/\mu_j, t_n = (z_j - v_j/\mu_j) + \sigma_{j,-}^n(x - x_j^n),$$

$$z + v/\mu_j, t_n = (z_j + v_j/\mu_j) + \sigma_{j,+}^n(x - x_j^n),$$

Fig. 2. Transmission and reflection of right going characteristic information $z - v/\mu_1$ at property interface.
for \( x_{j-1/2}^n \leq x \leq x_{j+1/2}^n \). The values at the cell centers \( x_j^* \) are still \( z_j^*, v_j^* \), and these are also the cell-averaged values. Reasonable choices for the slope, \( \sigma_{j,+} \), for example, would be

\[
\begin{align*}
\text{(a)} & \quad \frac{Q_{j,+}^n - Q_{j-1,+}^n}{x_{j+1}^n - x_{j-1}^n} = \frac{(z + v/\mu)_j^n - (z + v/\mu)_{j-1}^n}{x_{j+1}^n - x_{j-1}^n}, \\
\text{(b)} & \quad \frac{Q_{j,+}^n - Q_{j,+}^{n-1}}{x_{j+1}^n - x_{j}^n} = \frac{(z + v/\mu)_j^n - (z + v/\mu)_{j+1}^n}{x_{j+1}^n - x_{j}^n}, \\
\text{(c)} & \quad \frac{Q_{j+1,+}^n - Q_{j,+}^n}{x_{j+1}^n - x_{j}^n} = \frac{(z + v/\mu)_{j+1}^n - (z + v/\mu)_j^n}{x_{j+1}^n - x_{j}^n}.
\end{align*}
\]

Once the slopes \( \sigma_j^\pm \) are chosen and the data is reconstructed, the system (16), (17) is solved to obtain the interface values necessary to compute the flux integrals in (12) and (13):

\[
\int z = \frac{\mu_{j+1}}{\mu_j + \mu_{j+1}} \left[ Q_{j+1,+}^n - \frac{1}{2} \Delta x_j \sigma_{j+1,+}^n + \frac{1}{2} \Delta t \sigma_{j+1,+}^n (a_{j+1} + V) \right] \\
+ \frac{\mu_j}{\mu_j + \mu_{j+1}} \left[ Q_{j,-}^n + \frac{1}{2} \Delta x_j \sigma_{j,-}^n - \frac{1}{2} \Delta t \sigma_{j,-}^n (a_j - V) \right], \\
\int v = \frac{\mu_{j+1}}{\mu_j + \mu_{j+1}} \left[ Q_{j+1,+}^n - \frac{1}{2} \Delta x_j \sigma_{j+1,+}^n + \frac{1}{2} \Delta t \sigma_{j+1,+}^n (a_{j+1} + V) \right] \\
- \frac{\mu_j}{\mu_j + \mu_{j+1}} \left[ Q_{j,-}^n + \frac{1}{2} \Delta x_j \sigma_{j,-}^n - \frac{1}{2} \Delta t \sigma_{j,-}^n (a_j - V) \right].
\]

Second-order schemes have a dispersive property which produces overshoots and undershoots in regions of rapid transition of the solution variables. A considerable amount of work has gone into developing a remedy for this problem. One of the most important step in reaching this goal has been to incorporate into the numerical scheme the analytical result that the total variation of weak solutions to the scalar equation do not grow in time. This yielded a class of methods known as total variation diminishing (TVD) [8] schemes. ‘Limiters’ [24] are used to control the gradients of the reconstructed numerical solution so that the variation of the solution at the subsequent time step is no greater than that at the current time level. TVD schemes are second order accurate in smooth solution regions and, at solution extrema and in regions with steep solution gradients, they essentially revert to their first order version. The result is a higher-order scheme with overall accuracy greater than one, which does not produce numerical oscillations, and which still gives sharp overall resolution. For hyperbolic systems, one can construct a higher-order scheme by reconstructing the characteristic data as piecewise linear, and limiting these slopes so that the scheme is TVD in its characteristic variables.

The dispersive quality of purely second order schemes for hyperbolic differential equations with discontinuous flux functions is evident in the numerical solutions. See Fig. 3. However, these equations do not come with much (if any) supporting theory about the behaviour of the solutions that can immediately act as a guide in developing a means of controlling the appearance of spurious oscillations. For example, the total variation of the characteristic data often grows due to the partial transmission and reflection at material interfaces. See Fig. 4.
In the tests that we have conducted so far, we have found that the \textit{minmod} reconstructed slope gives a stable, higher-order scheme for the problem \((\rho \frac{\partial z}{\partial t})_x - (kz_x)_x = 0\), with piecewise constant property pattern of \(\rho\) and \(k\) as described earlier. The \textit{minmod} function is defined as
\[
\text{minmod}(x, y) = \begin{cases} 
0, & xy \leq 0, \\
\min(x, y), & 0 < x, y, \\
\max(x, y), & 0 > x, y.
\end{cases}
\]

The slope \( \sigma_{j}^{\pm} \) of the profile of \( z_{j}^{\pm} v/\mu = Q_{j}^{\pm} \) in cell \( j \) is

\[
\sigma_{j}^{\pm} = \text{minmod}\left\{ \frac{(Q_{j+1}^{\pm} - Q_{j}^{\pm})}{x_{j+1} - x_{j}}, \frac{(Q_{j}^{\pm} - Q_{j-1}^{\pm})}{x_{j} - x_{j-1}} \right\}.
\]

Thus, the slope is set to zero if the cell average of the associated variable \( Q^{+} \) or \( Q^{-} \) reaches an extremum in that cell, and is otherwise chosen to be the smaller of slope choices (b) and (c) that were listed earlier. Other choices of slope limiters, such as the Superbee limiter [24], produce unstable schemes. In subsequent research, we will investigate the source of instability/stability of the various schemes.

4. Numerical results

In this section, we present the results of computational experiments. For all of our test problems, the use of the limiting technique described in the previous section has yielded good results.

4.1. Wave motion through a static laminate

In [22], the authors use a Bloch expansion to obtain an effective medium description of wave propagation through a static (\( V = 0 \)) laminate which exhibits dispersion for large time. We use our numerical method to verify this theory for the one-dimensional problem. A similar investigation has been carried out in [5,6]. The problem controls are taken as

\[
(k_{1}, \rho_{1}) = (1, 1), \quad (k_{2}, \rho_{2}) = (3, 3), \quad m_{1} = 0.5,
\]

where \( m_{1} \) is the volume fraction of material 1. Both materials have characteristic speeds equal to 1, and we choose \( \Delta x_{1} = \Delta x_{2} = \Delta t \), so that the CFL number in each material region is unity. For a linear first-order method, this implies that there will be no numerical diffusion, and that the problem with piecewise constant initial data is solved exactly by the first-order upwind method. In particular, we take an initial Gaussian profile, discretized on a grid with \( \Delta x_{1} = \Delta x_{2} = 0.0005 \). A single material pair occupies a region of width 0.02. That is, the period \( \varepsilon \) is 0.02. We allow this profile to move through the laminate, via the first-order method, and run the wave problem until the solution separates into two distinct waves moving with opposite velocities. We take the right-going wave solution as the initial data for our experiment. See Fig. 5. A scaled, repeated Heaviside function is plotted along with the solution to indicate the microscale variation of the material parameters, relative to the wavelength of the disturbance.

In our experiment, the solution using the first-order upwind method with CFL numbers equal to 1 gives the exact solution. We compute with the second-order method with \( \Delta t = 0.8\Delta x_{1} \), i.e., with CFL numbers 0.8, and compare the solution of this method with the exact solution. Figs. 6 and 7 show the numerical solution of \( z \) at times 10 and 19, respectively. The values of the averaged solution \( \langle z \rangle \) are obtained by averaging over sequences of \( \varepsilon /\Delta x_{1} = 40 \) values of \( z_{j}^{\varepsilon} \).

By measuring the distance over which the peak of the profile has moved from time 0 to time 10, we see that the averaged solution moves with speed predicted by the formula for the static problem

\[
\Omega = (1/k)^{-1}(\rho)^{-1} = 0.866.
\]
Fig. 5. Initial profile is a right-going pulse centered at 0.

Fig. 6. Numerical solution of $z$ at time 10 as computed by higher-order scheme with $\Delta t/\Delta x_i = 0.8$. 

We also see clearly the effects of dispersion as predicted by Santosa and Symes in [22] for large times.

4.2. Wave motion through a dynamic laminate

In a spatio-temporal composite where the uniform property speed $V$ is non-zero, Lurie has shown [13] that coordinated wave motion occurs when

$$\tilde{k} \left( \frac{1}{\rho} \right) > V^2 > \frac{1}{\tilde{\rho}(1/k)},$$

where $\tilde{\xi} = m_1\xi_2 + m_2\xi_1$. The effective velocities $\lambda_1, \lambda_2$ are of the same sign, and a portion of the domain is screened from the effects of the wave disturbance.

In Figs. 8 and 9, we see the result of propagating an initial Gaussian profile with zero initial velocity through a right laminate (i.e., a dynamic laminate with positive effective velocities):

$$z(x, 0) = e^{-5x^2}, \quad z_t(x, 0) = 0,$$

$$(k_1, \rho_1) = (1, 1), \quad (k_2, \rho_2) = (10, 9), \quad m_1 = 0.5, \quad V = 0.8.$$  \hspace{1cm} (27)

These controls satisfy the condition for coordinated wave motion. In the computational domain $-2 \leq x \leq 10$, there are 200 property pair layers, so the composite material has a spatial period of $\epsilon = 12/200 = 0.06$. Since the initial disturbance has a support of width at least 2, the wavelength of the disturbance is several times larger than that of the period of the medium, and the effects of homogenization should be apparent. From Eq. (5), the effective velocities should be 0.36378913 and 0.96095791. Fig. 8 shows
Fig. 8. Detailed and averaged solution at time 8 of initial Gaussian profile through right laminate: 
\((k_1, \rho_1) = (1, 1), (k_2, \rho_2) = (10, 9), m_1 = 0.5, V = 0.8\).

Fig. 9. Contour plot of averaged solution at time 8 of initial Gaussian profile through right laminate: 
\((k_1, \rho_1) = (1, 1), (k_2, \rho_2) = (10, 9), m_1 = 0.5, V = 0.8\).
Fig. 10. Contour plot of averaged solution at time 8 of initial Gaussian profile through left laminate: \((k_1, \rho_1) = (1, 1), (k_2, \rho_2) = (10, 9), m_1 = \frac{1}{3}, V = -0.7\).

The detailed and numerically averaged solutions at time 8. The peaks of the D’Alembert waves of the averaged solutions travel distances 2.9425 ± \(\varepsilon\) and 7.6825 ± \(\varepsilon\) in time 8, putting the velocities from the numerical computations at 0.3678125 ± 0.0075 and 0.9603125 ± 0.0075. These numbers agree quite well with the theoretical speeds predicted by Lurie.

Fig. 9 is a contour plot showing the evolution of the averaged solution from time 0 to time 8. It is clear that the region \(x < -1\) is left relatively undisturbed by the effects of the original disturbance.

Fig. 10 is a contour plot illustrating the results of the numerical method as the initial profile propagates through a left laminate (i.e., a dynamic laminate with negative effective velocities):

\[(k_1, \rho_1) = (1, 1), \quad (k_2, \rho_2) = (10, 9), \quad m_1 = \frac{1}{3}, \quad V = -0.7.\]

The numerical results put the effective wave velocities in the ranges \(-0.9374999875 \pm 0.0075\) and \(-0.134999875 \pm 0.0075\), each respectively bracketing the theoretical wave velocities of \(-0.93949637\) and \(-0.13107872\). Fig. 10 clearly shows that the region \(x > 1\) is left relatively undisturbed by the effects of the original disturbance.

4.3. Comparison of numerical results with theory

In this subsection, we compare the results obtained via direct numerical solution of the propagation of disturbances through dynamic laminates with that predicted from analytical study.
Along with the standard homogenization technique [1], the analysis of equation (1) with the coefficients periodic in the argument \(x - Vt\) can be carried out with Floquet theory [3,23]. When the initial data is given in terms of the primary and dual variables, \(z\) and \(v\),

\[ z(x, 0) = z_0(x), \quad v(x, 0) = v_0(x), \]

the effective solution can be shown to be

\[ z(x, t) = \frac{1}{2} [z_0(x - \lambda_1 t) + z_0(x - \lambda_2 t)] + \frac{1}{2} \frac{\chi}{\mu_1 \theta_1 + \mu_2 \theta_2} [v_0(x - \lambda_2 t) - v_0(x - \lambda_1 t)], \tag{28} \]

up to a first approximation. Here

\[ \chi = \sqrt{\theta_1^2 + \theta_2^2 + 2\theta_1 \theta_2}, \quad \theta_i = \frac{m_i a_i}{V^2 - a_i^2}, \quad \sigma = \frac{\mu_1^2 + \mu_2^2}{2\mu_1 \mu_2}, \]

and the effective speeds \(\lambda_1, \lambda_2\) are given by (5). The solution (28) solves (4), the long wave approximation to a low frequency passing band.

Figs. 11–13 show the results for \(\langle z \rangle\) from the direct numerical solution on a fine grid with \(\Delta x_1 = \Delta x_2 = 1/240\), along with the theoretical solution (28). The initial data is \(z(x, 0) = e^{-5x^2}\), \(v(x, 0) = 0\) and the parameter data is given in (27). The period of the laminate is \(\epsilon = 0.05\), and the wavelength of the initial perturbation is roughly equal to 2. The values of the numerically averaged solution \(\langle z \rangle\) are obtained by averaging over sequences of \(e/\Delta x_1 = 12\) values of \(z_j^0\).

In the upper plots, the two D'Alembert wave profiles are clearly shown. The numerical and theoretical solutions match quite well—it is almost impossible to distinguish the two solutions for the faster

Fig. 11. Numerical solution and theoretical solution for \(\langle z \rangle\) at time 5: \((k_1, \rho_1) = (1, 1), (k_2, \rho_2) = (10, 9), m_1 = 0.5, V = 0.8\).
Fig. 12. Numerical solution and theoretical solution for $\langle z \rangle$ at time 15: $(k_1, \rho_1) = (1, 1)$, $(k_2, \rho_2) = (10, 9)$, $m_1 = 0.5$, $V = 0.8$.

Fig. 13. Numerical solution and theoretical solution for $\langle z \rangle$ at time 20: $(k_1, \rho_1) = (1, 1)$, $(k_2, \rho_2) = (10, 9)$, $m_1 = 0.5$, $V = 0.8$. 
moving wave. In the lower plots, we give a closer view of the slower wave. Here, we see that the numerical solution leads the theoretical profile by at least $\epsilon$. By time 15, one begins to see the effects of dispersion \[22\]. This higher-order effect is not captured by the first-order theoretical prediction given by (4) and (28).

4.4. Convergence study

Fig. 14 shows a log–log plot of the $L_1$ and $L_2$ errors in the solution of the wave propagation problem through a composite with a spatial period of $\epsilon = 0.064$. The initial data is

$$z(x, 0) = e^{-5x^2}, \quad z_t(x, 0) = 0,$$

and the problem parameters are

$$(k_1, \rho_1) = (1, 1), \quad (k_2, \rho_2) = (4, 4), \quad m_1 = 0.5, \quad V = 0, \quad (29)$$

We take the exact solution to the problem to be the numerical solution on a refined grid of cell width $\Delta x = 0.0005$. Since the wave speeds in the materials are equal ($a_1 = a_2 = 1$), we are able to compute with a CFL number of 1 in each cell to ensure the greatest accuracy possible. For the convergence analysis, the solution is computed on coarser grids of cell widths 0.001, 0.002, 0.004 and 0.008 with a CFL number of 0.9. We record the errors at time 2 after the initial profile has completely separated into left and right going D’Alembert waves. We find that the $L_1$ order of accuracy is 1.13 and the $L_2$ order is 1.0. On smooth solutions, one would expect the orders of accuracy to be closer to 2 rather than to 1. However,
though the individual materials have the same speed, their impedances $\mu = \sqrt{\rho k}$ differ. As a result, there is partial reflection and transmission of information at the interfaces giving a rapidly varying, non-smooth solution.

Fig. 15 compares the refined solution with the solution on the coarsest grid ($\Delta x = 0.001$).

5. Concluding remarks

In this paper, we presented a higher-order finite-volume method to model wave behaviour through a one-dimensional dynamic laminate. We used a grid moving with the velocity of the property pattern and used numerical formula which relied on the resolution of the propagating characteristic data. Using the minmod limiter gave a stable, higher order scheme which we have successfully used to carry out preliminary computational investigations of this new material. It is still necessary, however, to investigate the stability and instability of numerical schemes for the discontinuous coefficient wave problem.

We have been able to use the numerical experiments to verify the effective wave velocities predicted by Lurie, and to illustrate and validate his prediction of the screening effect. We also clearly see the method giving long time dispersive wave effects as follows from the analysis of Santosa and Symes for the static problem.

This form of direct simulation is important as we seek to understand the physics that leads to coordinated wave motion. We are in the process of investigating boundary effects in semi-infinite and finite domains, and we are also studying the supersonic material ($|V| > a_i$) which, from our initial studies, demonstrates very interesting results.
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Appendix A. Subsonic reflection and transmission coefficients

Consider the infinite elastic bar with stiffness, \( k \), and density, \( \rho \), given by

\[
(k, \rho) = \begin{cases} 
(k_1, \rho_1), & x < Vt, \\
(k_2, \rho_2), & x > Vt.
\end{cases}
\]

We investigate the reflected and transmitted waves that result when an incident wave encounters the interface \( x = Vt \) moving with speed \( |V| < a_1, a_2 \). The analysis is carried out for the case of a disturbance \( s(x + a_2 t) \), initially located to the right of the property discontinuity, moving through medium 2 into the interface.

The initial data is

\[
\begin{align*}
    z_1(x, 0) &= 0, \\
    \frac{\partial z_1}{\partial t}(x, 0) &= 0, & \text{for } x < 0, \\
    z_2(x, 0) &= s(x), \\
    \frac{\partial z_2}{\partial t}(x, 0) &= a_2 s'(x), & \text{for } x > 0.
\end{align*}
\]  

The solution of the wave equation in material 2 only, with the initial data above, is a wave traveling to the left with speed \( a_2 \), and for small \( t \), we assume that this is the only disturbance in the region to the right of the moving discontinuity. As \( t \) increases, however, other waveforms arise as the initial disturbance encounters the interface.

The general solution to the initial value problem is

\[
\begin{align*}
    z_1(x, t) &= f_1(x - a_1 t) + g_1(x + a_1 t), & x < Vt, \\
    z_2(x, t) &= f_2(x - a_2 t) + g_2(x + a_2 t), & x > Vt.
\end{align*}
\]  

From Eqs. (A.1) and (A.2), we have

\[
\begin{align*}
    0 &= f_1(x) + g_1(x), & x < 0, \\
    s(x) &= f_2(x) + g_2(x), & x > 0, \\
    0 &= f_1'(x) - g_1'(x), & x < 0, \\
    s'(x) &= -f_2'(x) + g_2'(x), & x > 0.
\end{align*}
\]  

Differentiating (A.5) and (A.6) yields

\[
\begin{align*}
    0 &= f_1'(x) + g_1'(x), & x < 0, \\
    s'(x) &= f_2'(x) + g_2'(x), & x > 0.
\end{align*}
\]  

Eqs. (A.7) and (A.9), then give \( g_1'(x) = 0 \) for \( x < 0 \), hence

\[
g_1(x) = 0 \quad \text{for } x < 0,
\]
and similarly,
\[ f_2(x) = 0 \quad \text{for } x > 0, \]
from (A.8) and (A.10). So, from (A.5) and (A.6),
\[ f_1(x) = 0, \quad x < 0, \quad g_2(x) = s(x), \quad x > 0. \]  
(A.11)  
(A.12)

The matching conditions across the moving interface $x = Vt$ are continuity of $z$ along interface $x = Vt$:
\[ z_1(Vt, t) = z_2(Vt, t), \]  
(A.13)

and
\[ \left( k_1 \frac{\partial z_1}{\partial x} + V \rho_1 \frac{\partial z_1}{\partial t} \right) (Vt, t) = \left( k_2 \frac{\partial z_2}{\partial x} + V \rho_2 \frac{\partial z_2}{\partial t} \right) (Vt, t), \]  
(A.14)

for $t > 0$. We note that the second condition is equivalent to continuity of the dual variable $v$ along the interface.

In terms of the functions $f_i, g_i$, these conditions are
\[ f_1(Vt - a_1) + g_1(Vt + a_1) = f_2(Vt - a_2) + g_2(Vt + a_2) \]

and
\[ (k_1 - V \rho_1 a_1) f'_1((V - a_1)t) + (k_1 + V \rho_1 a_1) g'_1((V + a_1)t) \]
\[ = (k_2 - V \rho_2 a_2) f'_2((V - a_2)t) + (k_2 + V \rho_2 a_2) g'_2((V + a_2)t). \]

By (A.11) and (A.12) and the assumption $|V| < a_1, a_2$, we arrive at a system of equations in the functions $g_1$ and $f_2$:
\[ g_1((V + a_1)t) - f_2((V - a_2)t) = s((V + a_2)t), \]  
(A.15)

\[ \mu_1(a_1 + V) g'_1((V + a_1)t) - \mu_2(a_2 - V) f'_2((V - a_2)t) = \mu_2(a_2 + V) s'((V + a_2)t). \]

Here we have used $\mu = \sqrt{k \rho} = \rho a = k/a$. We integrate this last equation with respect to $t$ to produce
\[ \mu_1 g_1((V + a_1)t) + \mu_2 f_2((V - a_2)t) = \mu_2 s((V + a_2)t). \]  
(A.16)

Solving system (A.15), (A.16) then gives
\[ g_1((V + a_1)t) = T s((V + a_2)t), \]
\[ f_2((V - a_2)t) = R s((V + a_2)t), \]

where
\[ T = \frac{2\mu_2}{\mu_2 + \mu_1}, \quad R = \frac{\mu_2 - \mu_1}{\mu_2 + \mu_1}. \]

Therefore,
\[ g_1(\omega) = T s \left( \omega \frac{(V + a_2)}{(V + a_1)} \right), \]
\[ f_2(\omega) = R s \left( \omega \frac{(V + a_2)}{(V - a_2)} \right), \]
for $\omega > 0$. 

Introducing these results into Eqs. (A.3) and (A.4) and using (A.11) and (A.12) gives the solution

\[ z_1(x, t) = Ts \left( (x + a_1t) \frac{V + a_2}{V + a_1} \right), \quad x < Vt, \tag{A.17} \]

\[ z_2(x, t) = s(x + a_2t) + Rs \left( (x - a_2t) \frac{V + a_2}{V + a_1} \right), \quad x > Vt. \tag{A.18} \]

On reaching the interface, the incident wave \( s \) produces two additional waves: the reflected wave moving to the right back into material 2, and the transmitted wave traveling to the left into material 1. \( R \) and \( T \) are the reflection and transmission coefficients that measure the amplitude of the resulting waveforms relative to that of the incident one.

References


