Wave Propagation and Energy Exchange in a Spatio-Temporal Material Composite with Rectangular Microstructure

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Abstract

We consider propagation of waves through a spatio-temporal doubly periodic material structure with rectangular microgeometry in one spatial dimension and time. Both spatial and temporal periods in this dynamic material are assumed to be of the same order of magnitude. A “double Floquet” solution is obtained in the special case when the wave equation \((pu_t)_t - (ku_z)_z = 0\) allows for the separation of variables. We also consider a checkerboard microgeometry where variables cannot be separated. The squares in a space-time checkerboard are assumed to be filled with materials having equal impedance but different phase speeds. Within certain parameter ranges, we observe numerically the formation of distinct and stable limiting characteristic paths (“limit cycles”) that attract neighbouring characteristics after a few time periods. The average speed of propagation along the limit cycles remains the same throughout certain ranges of parameters of the microgeometry (the “plateau effect”). We formulate, as a hypothesis, the statement saying that a checkerboard structure is on a plateau if and only if it yields stable limit cycles.

A dynamic material is a thermodynamically open system, as it is involved in a permanent exchange of energy and momentum with the environment. Material assemblages that produce the limit cycles are special in this aspect. Specifically, to make a wave travel through such an assemblage, we find analytically that an external agent may need to supply infinite energy and this may be so regardless of the wave frequency. For spatio-temporal laminates, however, an accumulation of energy (parametric resonance) may emerge only for frequencies that are not too low relative to some characteristic frequency of the system.
Key words: spatio-temporal, rectangular microstructure, limit cycles, composite, dynamic grating

1 Introduction

In this paper, we continue to work in the novel paradigm of dynamic materials or spatio-temporal composites. These are formations assembled from materials which are distributed on a microscale both in space and time. This material concept takes into consideration inertial, elastic, electromagnetic and other material properties that affect the dynamic behavior of various mechanical, electrical and environmental systems. In static or non-smart applications, the design variables, such as material density and stiffness, yield force and other structural parameters are position dependent but invariant in time. When it comes to dynamic applications, we also need temporal variability in the material properties in order to adequately match the changing environment. To this end, in dynamic material design, dynamic materials will take up the role played by ordinary composites in static material design. A dynamic disturbance on a scale much greater than the scale of a spatio-temporal microstructure may perceive this formation as a new material with its own effective properties. With spatio-temporal variability in the material constituents, we shall be able to effectively control the dynamic processes by creating effects that are unachievable through purely spatial (static) material design.

Dynamic materials may appear in very diverse physical implementations, most of them electromagnetic. We may distinguish two principal ways of making them by the spatio-temporal mixing of ordinary materials via the processes of activation and kinetization [1]. Activated dynamic materials are obtained by instantaneous or gradual change of the material parameters (stiffness, self-induction, capacitance, etc.) in various parts of the system in the absence of relative motion of those parts. Kinetic dynamic materials are obtained when various parts of the system are exposed to relative motion that is prearranged and generated in a certain way.

Dynamic materials rarely appear to be of natural origin (a living tissue being a notable exception). In order to make such materials work within a system, one should maintain the permanent exchange of energy and momentum between them and their environment; the whole system appears to be thermodynamically open. Therefore, dynamic materials should be artificially constructed, and such construction requires special technological solutions.

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A dynamic grating represents a clear example of an activated dynamic material. A grating is a material formation with spatially modulated optical properties. Laser systems with sufficient power can be used to modulate the optical properties of matter in real time [2]. Two intersecting light beams produce a dynamic grating in a material placed in the interference region. The grating spacing and amplitude are controlled by the intersection angle and beam power. Two excitation beams with different frequencies result in a propagating grating. Spatio-temporal variability of optical material properties may also be created in acoustooptic and magnetooptic devices [3].

From our ongoing mathematical and numerical study of these novel materials, though, it is clear that they hold a promising means of maintaining control over the properties of the material environment that conduct the dynamical disturbances. As a result, this class of materials will prove invaluable for various engineering and industrial applications.

With dynamic materials there are associated many unusual effects. For example, by appropriately controlling the design factors of an activated dynamic laminate, we find analytically and numerically that it is possible to selectively screen large domains in space-time from the invasion of long wave disturbances [4], [5]. With an ordinary (static) composite this screening effect is impossible. Also, in an ordinary waveguide, the waves with frequency below a certain critical cutoff value cannot travel. However, if the waveguide is filled with a dielectric characterized by a travelling periodic array of dielectric and magnetic properties, then, by a suitable choice of parameters of the array, the cutoff frequency can be eliminated. In other words, the waveguide may then support travelling waves of all frequencies.

These effects can be achieved by the simplest possible spatio-temporal microstructures – the laminates. To some extent, they appear to be special because they still allow for the standard procedure of homogenization and introduction of what is termed effective material parameters. However, not all spatio-temporal microstructures allow for this property.

In this paper, we consider a rectangular, doubly-periodic microstructure in one spatial dimension and time. The microscales in space and time are of the same order. In sections 2 and 3, we show that, under certain conditions, long waves may travel through such a microstructure, and we calculate the effective velocities of their propagation.

In sections 4 and 5, on the other hand, we present an example of a rectangular microstructure that does not allow for a homogenized description, at least not in its conventional version based on the assumption of finite energy stored in a system. A spatio-temporal checkerboard is such a structure. We show that some assemblages of this type are not transparent to long wave disturbances.
That is, in order for such disturbances to pass through the medium, one has to apply an infinite amount of energy. This is substantially different from the situation with laminates where non-transparency occurs only for waves that are sufficiently short. This property of checkerboard structures may be used to halt the transmission of low frequency waves along such devices as waveguides.

2 Rectangular microstructures in space-time

Spatio-temporal material composites or dynamic materials are composites in which the constituent properties such as permeability, permittivity, density, etc. are distributed on a microscale in time, as well as in space. Optimal material design for static or non-smart applications generally results in the formation of ordinary composites - mixtures where the design variables are position dependent but invariant in time. When it comes to dynamic applications, temporal variability in the material properties is also needed in order to adequately match the changing environment. Therefore, we have made it our goal to study such structures in order to understand their behaviour and potential applications. Spatio-temporal laminates are studied both analytically and numerically in [1], [4]–[10]. In this paper, we consider propagation of waves through a spatio-temporal material structure with rectangular microgeometry.

Consider a doubly-periodic material distribution in the \((z, t)\)-plane given by the pattern in Fig. 1. The rectangle \(-\ell_1 < z < \ell_2, -t_1 < t < t_2\) represents the basic cell of periodicity with periods \(\delta = \ell_1 + \ell_2\) in \(z\) and \(\tau = t_1 + t_2\) in \(t\). Rectangle \(i\) for \(i = 1, 2, 3, 4\) is occupied by a uniform material \(i\) having density \(\rho_i\) and stiffness \(k_i\). In an electromagnetic context, \(\rho_i\) and \(k_i\) would represent dielectric permittivity and the reciprocal of magnetic permeability. All materials are assumed immovable in a laboratory frame \(z, t\).

In this assemblage, we consider wave motion governed in each material by the linear second order equation

\[
(\rho u)_t - (k u_z)_z = 0, \quad (1)
\]

or, equivalently, by the system

\[
\rho u_t = v_z, \quad k u_z = v_t, \quad (2)
\]

with \(\rho, k\) taking values \(\rho_i, k_i\) within material \(i\). The waves pass from one material to another, maintaining the continuity of \(u\) and \(v\) across the interfaces separating the rectangles. The purpose of this paper is to study propagation of dynamic disturbances through such an assemblage. Both spatial and temporal periods, \(\delta\) and \(\tau\), will be assumed of the same order of magnitude, i.e.
\( \delta/\tau = O(a) \), where \( a = \sqrt{k/\rho} \) denotes the phase speed within any material constituent.

For each material, we have an elementary solution of (2),

\[
\begin{align*}
u &= \left( A e^{-\lambda^M_1} + Be^{\lambda^M_1} \right) \left( Ce^{-\lambda^M_1} + De^{\lambda^M_1} \right), \\
v &= \gamma \left( A e^{-\lambda^M_2} - Be^{\lambda^M_2} \right) \left( Ce^{-\lambda^M_1} - De^{\lambda^M_1} \right),
\end{align*}
\]

where \( a = \sqrt{k/\rho} \) and \( \gamma = \sqrt{k\rho} \) denote the phase velocity and wave impedance of the substance, respectively, and \( \lambda \) is a separation parameter.

Consider a layer \(-t_1 < t < 0 \) occupied by a \( \delta \)-periodic sequence of materials 1, 2 in \( z \) separated by vertical interfaces \( z = -\ell_1, \ z = 0, \ z = \ell_2 \), etc. as seen in Figure 1. Assume that the values \( \lambda, C, \) and \( D \) are constant along the layer; \( A \) and \( B \) satisfy the compatibility conditions

\[
A_{(1)} + B_{(1)} = A_{(2)} + B_{(2)}, \\
\gamma(1)(A_{(1)} - B_{(1)}) = \gamma(2)(A_{(2)} - B_{(2)}), \\
e^{\mu\delta} \left( A_{(1)} e^{\lambda_{a(1)}} + B_{(1)} e^{-\lambda_{a(1)}} \right) = A_{(2)} e^{-\lambda_{a(2)}} + B_{(2)} e^{\lambda_{a(2)}}, \\
\gamma(1)e^{\mu\delta} \left( A_{(1)} e^{\lambda_{a(1)}} - B_{(1)} e^{-\lambda_{a(1)}} \right) = \gamma(2) \left( A_{(2)} e^{-\lambda_{a(2)}} - B_{(2)} e^{\lambda_{a(2)}} \right),
\]

where the subscripts “(1)” and “(2)” relate to the relevant materials. The first pair of equations comes from the continuity conditions on \( z = 0 \). The second pair comes from continuity on \( z = \ell_2 \) where the Floquet relations

\[
u(z) = e^{\mu\delta}u(z - \delta), \quad v(z) = e^{\mu\delta}v(z - \delta),
\]

Fig. 1. Rectangular microstructure in \( z-t \).
are used to express solutions at \( z = \ell_2 \) in terms of the solutions at \( z = -\ell_1 \).

System (4) is satisfied if the Floquet exponent \( \mu \) takes one of two values \( \mu_{1,2} \) such that

\[
\mu_{1,2} \delta = \pm \chi(\theta_1, \theta_2),
\]

with \( \chi, \theta_1, \theta_2 \) defined by

\[
cosh \chi = \cosh \theta_1 \cosh \theta_2 + \sigma \sinh \theta_1 \sinh \theta_2,
\]

\[
\sigma = \frac{\gamma_{(1)}^2 + \gamma_{(2)}^2}{2 \gamma_{(1)} \gamma_{(2)}}, \quad \theta_1 = -\lambda \delta m_i/a(i),
\]

and

\[
m_1 = \ell_1/\delta, \quad m_2 = \ell_2/\delta.
\]

In the low frequency limit, \( |\lambda \delta/a(1)| \ll 1 \), we get [6]

\[
\mu_{1,2} = \pm \lambda \sqrt{\left\langle \frac{1}{k} \right\rangle_m \langle \rho \rangle_m}
\]

where

\[
\langle \xi \rangle_m = m_1 \xi_{(1)} + m_2 \xi_{(2)}
\]

denotes the arithmetic mean of \( \xi \).

The Floquet solution is given by the formulae [6]

\[
\begin{align*}
u & = [M_1 e^{\mu_1 z} P(\mu_1, z) + M_2 e^{\mu_2 z} P(\mu_2, z)] (Ce^{-\lambda t} + De^{\lambda t}), \\
v & = [M_1 e^{\mu_1 z} Q(\mu_1, z) + M_2 e^{\mu_2 z} Q(\mu_2, z)] (Ce^{-\lambda t} - De^{\lambda t}),
\end{align*}
\]

with \( P(\mu, z), Q(\mu, z) \) specified as

\[
P(\mu, z) = \begin{cases} 
- e^{-\left(\mu_+ a_{(1)}\right)(z-j\delta)} + \lambda e^{-\left(\mu_+ a_{(2)}\right)(z-j\delta)}, & (j - m_1)\delta < z < j\delta, \quad j = 1, 2, \ldots, \\
Ke^{-\left(\mu_+ a_{(2)}\right)(z-j\delta)} + Le^{-\left(\mu_+ a_{(2)}\right)(z-j\delta)}, & j\delta < z < (j + m_2)\delta, \quad j = 0, 1, 2, \ldots,
\end{cases}
\]

\[
Q(\mu, z) = \begin{cases} 
\gamma_{(1)} \left[ - e^{-\left(\mu_+ a_{(1)}\right)(z-j\delta)} + \lambda e^{-\left(\mu_+ a_{(1)}\right)(z-j\delta)} \right], & (j - m_1)\delta < z < j\delta, j = 1, 2, \ldots, \\
\gamma_{(2)} \left[ -Ke^{-\left(\mu_+ a_{(2)}\right)(z-j\delta)} + Le^{-\left(\mu_+ a_{(2)}\right)(z-j\delta)} \right], & j\delta < z < (j + m_2)\delta, j = 0, 1, 2, \ldots
\end{cases}
\]

Here \( \mu \) takes the values \( \mu_1, \mu_2 \), and \( I, K, L \) are solutions of the system

\[
\begin{align*}
-I + K + L &= 1, \\
I + (K - L)(\gamma_{(2)}/\gamma_{(1)}) &= 1, \\
-I e^{\theta_1} + Ke^{\theta_2 + \chi} + Le^{-\theta_2 + \chi} &= e^{-\theta_1},
\end{align*}
\]
with the upper (lower) sign of \( \mp \) related to \( \mu_1(\mu_2) \). Both \( P(\mu, z), Q(\mu, z) \) are \( \delta \)-periodic in \( z \). System (9) specifies the modulated waves with \( e^{\mu z} \) being the modulation factor and \( P(\mu, z), Q(\mu, z) \) representing the short wave carriers.

Consider the layer \( 0 < t < t_2 \). We observe that equations (3)–(12) remain valid for it as well with obvious modifications. The symbols \( A, \ldots, D, I, K, L, P, Q, \theta_1, \theta_2, \chi, \mu_1, \mu_2, \lambda \) should be replaced by the relevant symbols \( \tilde{A}, \ldots, \tilde{D}, \tilde{I}, \tilde{K}, \tilde{L}, \tilde{P}, \tilde{Q}, \tilde{\theta}_1, \tilde{\theta}_2, \tilde{\chi}, \tilde{\lambda} \), and material constants \( k, \rho, a, \gamma \) take values \( k_{(3)}, \ldots, \gamma_{(3)} \) and \( k_{(4)}, \ldots, \gamma_{(4)} \) in materials 3 and 4.

For the layer \( t_2 < t < t_2 + t_1 \), we apply equations (3)–(12) with \( A, \ldots, \lambda \) replaced by \( \tilde{A}, \ldots, \tilde{\lambda} \), and \( k, \ldots, \gamma \) taking values \( k_{(1)}, \ldots, \gamma_{(1)} \) and \( k_{(2)}, \ldots, \gamma_{(2)} \) in materials 1 and 2.

On the interface \( t = 0 \), we have compatibility conditions expressing the continuity of \( u \) and \( v \):

\[
[M_1 e^{\mu_1 z} P(\mu_1, z) + M_2 e^{\mu_2 z} P(\mu_2, z)](C + D) = [\tilde{M}_1 e^{\mu_1 z} \tilde{P}(\mu_1, z) + \tilde{M}_2 e^{\mu_2 z} \tilde{P}(\mu_2, z)](\tilde{C} + \tilde{D})
\]

\[
[M_1 e^{\mu_1 z} Q(\mu_1, z) + M_2 e^{\mu_2 z} Q(\mu_2, z)](C - D) = [\tilde{M}_1 e^{\mu_1 z} \tilde{Q}(\mu_1, z) + \tilde{M}_2 e^{\mu_2 z} \tilde{Q}(\mu_2, z)](\tilde{C} - \tilde{D}).
\]

(13)

A similar system holds on the interface \( t = t_2 \).

Clearly, equations (13) are satisfied only if the coefficients of \( C + D \) and \( \tilde{C} + \tilde{D} \), as well as of \( C - D \) and \( \tilde{C} - \tilde{D} \), are constant multiples of each other. It will be shown in the next section that this happens if the material layout represented in Figure 1 is such that the system (2) allows for separation of variables.

3 Case of separation of variables

The variables \( z, t \) are separated in (2) if \( \rho(z, t) \) and \( k(z, t) \) appear to be products of functions that depend on the single variable \( z \) and the single variable \( t \) alone:

\[
\rho = \rho^Z(z)\rho^T(t), \quad k = k^Z(z)k^T(t).
\]

(14)

We look for \( u(z, t) \), the solution of (1) in the form of a product \( u^Z(z)u^T(t) \). Then \( u^Z \) and \( u^T \) will be solutions of

\[
(k^Z u^Z_z)_z - \lambda^2 \rho^Z u^Z = 0, \quad (\rho^T u^T_t)_t - \lambda^2 k^T u^T = 0,
\]

(15)

with a separation constant \( \lambda \).
Assume now that each of the two pairs of functions $\rho^Z, k^Z$ and $\rho^T, k^T$ takes different values in the relevant base intervals of periodicity $-\ell_1 < z < \ell_2$, $-t_1 < t < t_2$:

$$
\rho^Z, k^Z = \begin{cases} 
\rho^Z_1, k^Z_1, & -\ell_1 < z < 0, \\
\rho^Z_2, k^Z_2, & 0 < z < \ell_2,
\end{cases} \quad \quad (16)
$$

$$
\rho^T, k^T = \begin{cases} 
\rho^T_1, k^T_1, & -t_1 < t < 0, \\
\rho^T_2, k^T_2, & 0 < t < t_2,
\end{cases} \quad \quad (17)
$$

In other words, we have the following characterization of materials 1, \ldots, 4 (see Fig. 1).

Material 1: $\rho(1) = \rho^Z_1 \rho^T_1$, $k(1) = k^Z_1 k^T_1$,

Material 2: $\rho(2) = \rho^Z_2 \rho^T_1$, $k(2) = k^Z_2 k^T_1$,

Material 3: $\rho(3) = \rho^Z_1 \rho^T_2$, $k(3) = k^Z_1 k^T_2$,

Material 4: $\rho(4) = \rho^Z_2 \rho^T_2$, $k(4) = k^Z_2 k^T_2$. 

(18)

Note that $\rho^Z_j, \rho^T_j$ and $k^Z_j, k^T_j$ for $j = 1, 2$ have the dimensions of the square roots of $\rho$ and $k$, respectively.

Equations (2) allow for the following elementary solutions,

$$
\begin{align*}
  u & = \left( A e^{-\lambda_a t} + B e^{\lambda_a t} \right) \left( C e^{-\lambda_a t} + D e^{\lambda_a t} \right), \\
  v & = \gamma^Z \left( A e^{-\lambda_a t} - B e^{\lambda_a t} \right) \gamma^T \left( C e^{-\lambda_a t} - D e^{\lambda_a t} \right),
\end{align*} \quad \quad (19)
$$

with symbols

$$
\begin{align*}
  a^Z & = \sqrt{k^Z / \rho^Z}, & a^T & = \sqrt{k^T / \rho^T}, & \gamma^Z & = \sqrt{k^Z \rho^Z}, & \gamma^T & = \sqrt{k^T \rho^T},
\end{align*}
$$

specified by (16) and (17). Note the relation between the symbols in this section and the phase velocities and wave impedances in Section 1:

$$
\begin{align*}
  a_{(i)} & = a^Z_{(i)} a^T_{(i)}, & \gamma & = \gamma^Z_{(i)} \gamma^T_{(i)}.
\end{align*}
$$

The values of $(a^Z_{(i)})^2, (a^T_{(i)})^2, (\gamma^Z_{(i)})^2$ and $(\gamma^T_{(i)})^2$ related to various materials are summarized in Table 1.

Consider the layer $-t_1 < t < 0$ occupied by a $\delta$-periodic sequence of materials 1 and 2. Referring to Table 1, we observe that $a^T$ and $\gamma^T$ are the same for both materials. The Floquet solutions for this layer are therefore specified by equations (6)–(9), with obvious modifications generated by equations (15). In particular, the long wave Floquet exponent $\mu$ in (7) becomes

$$
\mu_{1,2} = \pm \mu = \pm \lambda \sqrt{\left( \frac{m_1}{k^Z_1} + \frac{m_2}{k^Z_2} \right) \left( m_1 \rho^Z_1 + m_2 \rho^Z_2 \right)} = \pm \lambda \sqrt{\left( \frac{1}{k^Z} \right) m \langle \rho^Z \rangle_m}. \quad (20)
$$
<table>
<thead>
<tr>
<th>Material</th>
<th>((a^z)^2)</th>
<th>((a^T)^2)</th>
<th>((\gamma^z)^2)</th>
<th>((\gamma^T)^2)</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>(k_1^z/\rho_1^z)</td>
<td>(k_1^T/\rho_1^T)</td>
<td>(k_1^z/\rho_1^z)</td>
<td>(k_1^T/\rho_1^T)</td>
</tr>
<tr>
<td>2</td>
<td>(k_2^z/\rho_2^z)</td>
<td>(k_2^T/\rho_2^T)</td>
<td>(k_2^z/\rho_2^z)</td>
<td>(k_2^T/\rho_2^T)</td>
</tr>
<tr>
<td>3</td>
<td>(k_1^z/\rho_1^z)</td>
<td>(k_1^T/\rho_1^T)</td>
<td>(k_1^z/\rho_1^z)</td>
<td>(k_1^T/\rho_1^T)</td>
</tr>
<tr>
<td>4</td>
<td>(k_2^z/\rho_2^z)</td>
<td>(k_2^T/\rho_2^T)</td>
<td>(k_2^z/\rho_2^z)</td>
<td>(k_2^T/\rho_2^T)</td>
</tr>
</tbody>
</table>

Table 1

Values of \(a^z, a^T, \gamma^z, \) and \(\gamma^T\) related to materials 1, 2, 3, 4 in a rectangular microstructure.

These values do not depend on \(k_1^T\) and \(\rho_1^T\). The functions \(P(\mu, z), Q(\mu, z)\) in (9) have the structure given by equations (10), (11) with \(\gamma_{(1)}, \cdots, a_{(2)}\) replaced by \(\gamma_{(1)}, \cdots, a_{(2)}\), respectively. The solution (9) then becomes

\[
\begin{align*}
    u & = [M_1 e^{\mu_1 z} P(\mu_1, z) + M_2 e^{\mu_2 z} P(\mu_2, z)] \left( C e^{-\lambda \alpha_1^T t} + D e^{\lambda \alpha_1^T t} \right), \\
    v & = [M_1 e^{\mu_1 z} Q(\mu_1, z) + M_2 e^{\mu_2 z} Q(\mu_2, z)] \left( C e^{-\lambda \alpha_2^T t} - D e^{\lambda \alpha_2^T t} \right). 
\end{align*}
\] (21)

The factors in the square brackets represent the Floquet solutions related to the first equation (15) and generated by a \(\delta\)-periodic sequence of materials distributed along the \(z\)-axis and possessing properties \((\rho_1^z, k_1^z)\) and \((\rho_2^z, k_2^z)\). Equations (21) are related to the layer \(-t_1 < t < 0\); their structure is similar to that of (19). When we pass to the next layer \(0 < t < t_2\), the solution preserves this structure, the \(z\)-dependent factors in the square brackets remain the same, as seen from Table 1, whereas \(a_{(1)}^z\) gives way to \(a_{(2)}^z\), and \(\gamma_{(1)}^T\) is replaced by \(\gamma_{(2)}^T\). We now apply the Floquet procedure to a \(\tau\)-periodic sequence of layers perpendicular to the \(t\)-axis, and arrive at the final solution

\[
\begin{align*}
    u & = [M_1 e^{\mu_1 z} P(\mu_1, z) + M_2 e^{\mu_2 z} P(\mu_2, z)] \\
    & \quad \left[ N_1 e^{\nu_1 t} R(\nu_1, t) + N_2 e^{\nu_2 t} R(\nu_2, t) \right], \\
    v & = [M_1 e^{\mu_1 z} Q(\mu_1, z) + M_2 e^{\mu_2 z} Q(\mu_2, z)] \\
    & \quad \left[ N_1 e^{\nu_1 t} S(\nu_1, t) + N_2 e^{\nu_2 t} S(\nu_2, t) \right],
\end{align*}
\] (22)

with low frequency Floquet exponents

\[
\nu_{1,2} = \pm \nu = \pm \lambda \sqrt{(n_1 k_1^T + n_2 k_2^T) \left( \frac{n_1}{\rho_1^T} + \frac{n_2}{\rho_2^T} \right)} = \pm \lambda \sqrt{(k^T)\rho_n \left( \frac{1}{\rho^T} \right)_n} 
\] (24)

where

\[
n_1 = t_1/\tau, \quad n_2 = t_2/\tau.
\]

The \(\tau\)-periodic functions \(R(\nu, t), S(\nu, t)\) are specified by the expressions for \(P\) and \(Q\) in (10) and (11) with \(t\) used instead of \(z\), \(\nu\) instead of \(\mu\), \(a_{(1)}\) replaced by
\[(a(Z)_{(1)})^{-1}, a(2) \text{ by } (a(T)_{(2)})^{-1}, \gamma(1) \text{ replaced by } \gamma(T)_{(1)} \text{ and } \gamma(2) \text{ by } \gamma(T)_{(2)}; \] also, \(n_i\) should replace \(m_i\), and \(\tau\) replace \(\delta\).

By (20) and (24), we conclude that a general solution (22), (23) is a combination of modulated waves with envelopes
\[e^{\mu x + \nu t}\]

propagating, in the case of low frequency, with group velocities
\[
\pm \nu / \mu = \pm \sqrt{\left\langle k^T \right\rangle_n \left\langle \frac{1}{k^Z} \right\rangle_m^{-1} \left\langle \frac{1}{\rho^Z} \right\rangle_m^{-1} \left\langle \frac{1}{\rho^T} \right\rangle_n^{-1}}
\]

(25)
The factors \(\mu\) and \(\nu\) represent the Floquet exponents generated by the periodic dependency of the property pattern. The “double Floquet” behavior is a consequence of the ability to separate the variables in our problem.

In the next section, we examine another case of wave propagation through a rectangular material structure in space-time. Specifically, we choose a checkerboard assemblage made up of two materials having the same wave impedance. For this particular class of structures, we will be able to make some conclusions about the effective velocities of wave propagation.

4 Checkerboard assemblage of materials with equal wave impedance

We consider here a special case of the rectangular spatio-temporal material structure as represented in Figure 1. Suppose material 3 is the same as material 2, and material 4 is the same as material 1; we call such a layout a ‘checkerboard’. In addition, we will later assume that the two materials 1 and 2 have the same value of the wave impedance \(\gamma\). It is easy to see that the variables in (2) cannot be separated in this case.

Within a pure material, the general solution of (2) can be easily constructed from the values of the two Riemann invariants \(R = u - v / \gamma\) and \(L = u + v / \gamma\) which are respectively governed by the scalar advection equations,
\[
R_t + a R_z = 0,
\]

(26)
and
\[
L_t - a L_z = 0,
\]

(27)
with \(a = \sqrt{k / \rho}\) being the phase speed of the material. We look at an elementary solution
\[
u = -\gamma(1) A e^{\lambda (t - z / a(1))},
\]

(28)
representing a wave which travels through material 1 in a positive z-direction. When such a wave reaches an interface \( z = 0 \) separating material 1 from material 2, it splits into a reflected wave

\[
    u = \frac{\gamma(1) - \gamma(2)}{\gamma(1) + \gamma(2)} A e^{\lambda(t+z/a(1))}, \quad v = \gamma(1) \frac{\gamma(1) - \gamma(2)}{\gamma(1) + \gamma(2)} A e^{\lambda(t+z/a(2))},
\]

(29)

which heads to the left, back into material 1, and a transmitted wave

\[
    \bar{u} = \frac{2\gamma(1)}{\gamma(1) + \gamma(2)} A e^{\lambda(t-z/a(2))}, \quad \bar{v} = -\frac{2\gamma(1)}{\gamma(1) + \gamma(2)} A e^{\lambda(t-z/a(2))},
\]

(30)

which continues through the interface into material 2. When a wave (28) reaches a 'horizontal' interface \( t = 0 \) separating material 1 from material 2 though, two waves are generated which both move into material 2 [9]. The general solution is

\[
    \bar{u} = \frac{A}{2} \left[ \frac{\gamma(2) - \gamma(1)}{\gamma(2)} e^{-\bar{\lambda}(t+z/a(2))} + \frac{\gamma(2) + \gamma(1)}{\gamma(2)} e^{\bar{\lambda}(t-z/a(2))} \right],
\]

(31)

\[
    \bar{v} = \frac{A}{2} \left[ \left(\gamma(2) - \gamma(1)\right) e^{-\bar{\lambda}(t+z/a(2))} - \left(\gamma(2) + \gamma(1)\right) e^{\bar{\lambda}(t-z/a(2))} \right],
\]

(32)

where \( \bar{\lambda} = \lambda a(2)/a(1) \).

For our special checkerboard structure, we assume that materials 1 and 2 have the same wave impedance, \( \gamma(1) = \gamma(2) = \gamma \). In this case, there is no reflected wave (29), and there is only one transmitted wave in (31), (32). The incident wave (28) passes through interfaces undiminished in amplitude but with a change in frequency or wave number. In this and the next sections, we shall only consider waves propagating in the positive z-direction. Waves propagating in the negative z-direction will follow a similar analysis.

Our goal is to study and understand how disturbances propagate through this checkerboard microstructure. To do this, we simulate numerically wave motion through several material arrangements, and then make some conjectures based on our experimental observations. The units of space and time in the examples we present are chosen so that the periods of the assemblage along the z and t axes are both unity, that is, \( \delta = \tau = 1 \). When \( m_1 = 0 \) or \( m_1 = 1 \), we have a temporal laminate; if \( m_1 = 0 \) or 1, then this is a static laminate. Since wave impedance \( \gamma(i) = \sqrt{K(i)/\rho(i)} \) is assumed to be the same throughout the entire structure, we distinguish between the two constituent materials via their phase speeds \( a(i) = \sqrt{K(i)/\rho(i)} \). Without loss of generality, we take \( \gamma(1) = \gamma(2) = 1 \).

In our first experiment, we consider the structure with parameters \( m_1 = 0.4, n_1 = 0.5, a(1) = 0.6 \) and \( a(2) = 1.1 \). Figure 2 represents the paths of right-going disturbances (28) which originate on the interval \([0, 2]\) at time 0.
Fig. 2. Limit cycles in the checkerboard structure with $a_{(1)} = 0.6, a_{(2)} = 1.1, m_1 = 0.4, n_1 = 0.5$.

Time is measured along the vertical axis of this figure. The vertical and horizontal lines define the checkerboard arrangement. It is clear to see that within each period, the group of paths in Figure 2 separates into two distinct arrays that each converges to its own limiting path (“limit cycle”) after a few time periods. The limit paths are called cycles because the trajectory pattern cycles or repeats. Such cycles are parallel to each other and have a common average slope equal to 1. Each cycle is stable; it attracts trajectories which originate on the initial manifold at the left and the right of the point of origination of the cycle itself. In the example given, the cycles originate around $z = 0.5$ and $z = 1.5$ at time 0, and are indicated by the paths in bold. There is one limit cycle per spatial period. Successive stable limit cycles are separated by an unstable limit cycle. After close numerical inspection, we find that such unstable cycles originate, at time 0, at points $n + 0.375$ for integers $n$, and at $n + 0.4953$ for stable limit cycles.

This convergence phenomenon manifests itself through concentration of the initial disturbance, and is illustrated in the solution profile sequence of Figure 3. The vertical axis is $u$, and $z$ is on the horizontal axis. The profiles are computed from system (2) via a finite volume scheme which is a blend of the techniques used in [5] and [9]. The initial disturbance is a Gaussian; we may regard it as having support on $[0.5, 1.5]$. We show evolution profiles up to time 3; the speed of the disturbance is seen to be 1. As the disturbance travels through the checkerboard material, the information that was initially spread over the region $[0.5, 1.35]$ has, roughly speaking, by time 3, concentrated within the narrower region $[3.5, 3.65]$. The data is compressed as expected by
the trajectory behaviour illustrated in Figure 2. The information that was initially associated with $z$ values in $[1.35, 1.37]$ has, by time 3, been spread over the interval $[3.65, 4.4]$ giving an almost constant state, while the rest of the solution changes more rapidly over $[4.4, 4.5]$.

In Figure 4, we plot the solution at time 20 of a Gaussian disturbance with support about 10 times wider than that in Figure 3, which has gone through the same checkerboard structure as above. The solution is piecewise constant taking values of the initial data at $z = n + 0.37$ for $n = -5, \cdots, 5$. To see this, we also plot the initial data shifted to the right 20 units. The constant states occupy a space interval of length $\delta = 1$ since there is only one stable limit cycle per period.

![Graphs showing disturbance evolution](image)

**Fig. 3.** Evolution of a disturbance through a structure with $m_1 = 0.4, n_1 = 0.5, a_{(1)} = 0.6$ and $a_{(2)} = 1.1$.

Next, we consider the structure with the same values of $a_i, m_i$ as before but with $n_1 = 0.8$. Unlike the first structure, the paths in Figures 5 and 6 do not demonstrate stable convergence to isolated asymptotic routes. Instead, the trajectories engage in a regular pattern of drift towards and then away from would-be limit cycles. This trend is periodic and the wavelength of this pattern is about 10 times the period of the structure itself. From the trajectories, we compute that the average speed of the disturbances is roughly 0.9.

If we reduce $n_1$ to 0.1, we see very little remnants of the existence of limit cycles. The wave trajectories more or less occupy the entire strip. See Figure 7. The average asymptotic speed of these paths is roughly 0.77.

The four parameters $a_{(1)}, a_{(2)}, m_1, n_1$ determine the checkerboard material,
Fig. 4. Solution at time 10 of a disturbance with wide support through a structure with $m_1 = 0.4, n_1 = 0.5, a_{(1)} = 0.6, a_{(2)} = 1.1$, and initial data shifted right 10 units.

Fig. 5. Low frequency pattern in trajectories through structure with $m_1 = 0.4, n_1 = 0.8, a_{(1)} = 0.6$ and $a_{(2)} = 1.1$.

and hence determine the manner in which disturbances travel through such structures. In the three examples presented above, $a_{(1)}, a_{(2)}$ and $m_1$ were fixed, and by varying the value of $n_1$ only, we are able to see different trajectory behaviour and different average speeds. In Figure 8, we plot graphs of average speed versus $n_1$ for a sequence of $m_1$ values. Define the speed in the structures as $f(m_1, n_1)$. Notice that $f(m, n) = f(1 - m, 1 - n)$. This is so because, in space-time, each period of the structure with volume fractions $(m, n)$ is made up of an $m \times n$ and an $(1 - m) \times (1 - n)$ rectangles of material 1, and the rest is filled with material 2. Thus, the checkerboard structure with volume
Fig. 6. Closer view of wave trajectories through structure with \( m_1 = 0.4, n_1 = 0.8, a_1 = 0.6 \) and \( a_2 = 1.1 \).

Fig. 7. Structure with \( m_1 = 0.4, n_1 = 0.1, a_1 = 0.6 \) and \( a_2 = 1.1 \) fractions \((m, n)\) is the same as that with volume fractions \((1 - m, 1 - n)\).

In several of the plots, we see intervals of \( n_1 \) for which \( f(m_1, n_1) \) is constant for a given \( m_1 \) value; we call these “plateaux” and refer to the associated structures as “being on a plateau”. By inspecting the plots in Figure 8, it is seen that for \( a_1 = 0.6 \) and \( a_2 = 1.1 \), there are always plateaux corresponding to a speed equal to unity. In the first example of this section where we observed the existence of stable limit cycles, we had \((m_1, n_1) = (0.4, 0.5)\). The propagation speed in such a structure is \(1 = f(0.4, 0.5)\), and this material puts us on the plateau of the fourth plot of the series shown in Figure 8. The other structures shown in Figures 6 and 7 are not on a plateau and do not exhibit limit cycles.
Fig. 8. Wave speeds as a function of $m_1$ and $n_1$ for $a_1 = 0.6$ and $a_2 = 1.1$.

Fig. 9. Trajectories in material with $a_1 = 0.6, a_2 = 1.1, n_1 = 0.4$ and $n_1$ as indicated.

Figure 9 gives portions of trajectories which originate on $[0, 1]$ at time 0 in twelve checkerboard structures distinguished only by their values of $n_1$. The other parameter values are $a_1 = 0.6, a_2 = 1.1, n_1 = 0.4$. By comparing the values of $n_1$ which yield limit cycles with the location of the plateau in the velocity-$n_1$ graph for $m_1 = 0.4$ in Figure 10, we propose the following hypothesis:
Hypothesis 1 *A structure is on a plateau if and only if the structure yields stable limit cycles.*

In Figures 11 and 12, we see how speeds vary with $n_1$ for distinct values of $a(2)$, with $a(1) = 0.6, m_1 = 0.4$. Note that $a(2) = 1$ is a crucial case, since there will always be a trajectory that moves with constant speed $1 = \delta / \tau$ because it passes through the corners of the checkerboard so as to remain always in material 2 and never be deflected by entering material 1. Furthermore, when $a(1), a(2) < 1$, there are no limit cycles with speed 1.

The limit cycle to which an array of trajectories converges is such that if it passes through the point $(z, t)$ in the $z-t$ plane, then it also passes through the point $(z + q\delta, t + p\tau)$ for some integers $p, q$. We take the speed of travel to be $\frac{z}{p\tau}$. So, in our computed examples, the speeds should be rational numbers since $\delta = \tau = 1$. Figure 13 suggests that for $a(1) = 0.6, a(2) = 0.9, m_1 = 0.15$ there are at least 2 clear plateaux (maybe 3) indicating values of $n_1$ for which limit cycles have rational speeds. Figures 14 and 15 in which $n_1$ takes the
values of 0.55 and 0.2, respectively, support the observation that there are limit cycles for the structures on the plateaux and that the associated cycle speeds are 3/4 and 2/3.

Figure 16 shows the solution at time 30 of a disturbance through the checker-board structure with parameters \( a_{(1)} = 0.6, a_{(2)} = 0.9, m_1 = 0.15, n_1 = 0.2 \). The initial data shifted to the right 20 units is also shown. Compare this figure to Figure 4. The trajectory paths in Figure 15 show that there are 3 stable limit cycles per period and so we see that the piecewise constant solution consists of 3 constant states per spatial period \( \delta = 1 \). In general, for a structure on a plateau, the asymptotic solutions in the limit \( t \to \infty \) for non-zero values of the ratio of the period of the structure to the characteristic wavelength of the disturbance are discontinuous. However, when this ratio approaches zero, the solution generated by continuous initial data tends to become continuous for any finite \( t \).

Fig. 13. Limit cycles have speeds that are rational multiples of \( \delta/\tau = 1 \). Here, \( a_{(1)} = 0.6, a_{(2)} = 0.9, m_1 = 0.15 \).
Fig. 14. Wave speed = 3/4 when $a_1 = 0.6, a_2 = 0.9, m_1 = 0.15$ and $n_1 = 0.55$.

Fig. 15. Wave speed = 2/3 when $a_1 = 0.6, a_2 = 0.9, m_1 = 0.15$, and $n_1 = 0.2$.

Figures 17, 18 and 19 have randomly generated values for $a_1, a_2, m_1, n_1$ which give limit cycles. The limit cycles travel at rational speeds as expected by our hypothesis.

These observations are in accordance with Poincaré’s Theorem indicating the existence of the average speed termed the rotation number in Poincaré’s formulation. It is known [11] that this speed is rational if and only if the phase curve of the differential equation

$$\frac{dz}{dt} = a$$

is closed on the torus. At the same time, this rational value of rotation number persists over a range of structural parameters giving what we have called plateaux. This range can be wide enough, thus securing stability of rational rotation numbers.
Solution at time 30

Initial data shifted

Fig. 16. Solution at time 30 of a disturbance with wide support through a structure with \(a_1 = 0.6, a_2 = 0.9, m_1 = 0.15, n_1 = 0.2\) and initial data shifted right 20 units.

Wave speed = 1/2. Use \(m_1 = 0.0579, n_1 = 0.3529, a_1 = 0.8132, a_2 = 0.0099\) (randomly generated parameters)

5 Energy transformation in the presence of limit cycles

The formation of limit cycles illustrated in Figure 2 is accompanied by a special energy/momentum exchange between the dynamic material and the environment. A closer look, as in Figure 20, reveals interesting behaviour of characteristics that are close to the limit cycle: they always enter material 1 and leave material 2 across a vertical interface, and leave material 1 and enter material 2 across a horizontal interface. As a result of this special kinematics,
Fig. 18. Wave speed = 2/7. Use $m_1 = 0.8757$, $n_1 = 0.7373$, $a_{(1)} = 0.4096$, $a_{(2)} = 0.0353$ (randomly generated parameters)

Fig. 19. Wave speed = 2/5. Use $m_1 = 0.5651$, $n_1 = 0.9692$, $a_{(1)} = 0.1187$, $a_{(2)} = 4.3511$ (randomly generated parameters)

a group of parallel characteristics gains some finite portion of energy from the outside agent each time it enters material 2. This happens because, at this moment, an external agent performs a finite amount of work against the inertial and elastic forces.

Consider the energy equation

$$\frac{\partial}{\partial t} W_{tt} + \frac{\partial}{\partial z} W_{tz} = -\frac{1}{2} \left( \frac{\partial \rho}{\partial t} \left( \frac{\partial u}{\partial t} \right)^2 - \frac{\partial k}{\partial t} \left( \frac{\partial u}{\partial z} \right)^2 \right).$$  \hspace{1cm} (33)

Here,

$$W_u = \frac{1}{2} \rho \left( \frac{\partial u}{\partial t} \right)^2 + \frac{1}{2} k \left( \frac{\partial u}{\partial z} \right)^2$$
Fig. 20. A closer look at limit cycles in the checkerboard structure with $a_{(1)} = 0.6, a_{(2)} = 1.1, m_1 = 0.4, n_1 = 0.5$.

is the energy density, and

$$W_{t z} = -k \frac{\partial u}{\partial t} \frac{\partial u}{\partial z}$$

is the energy flux density. Integrate (33) over a narrow horizontal strip

$$z_0 < z < z_1, \quad t_* - \beta < t < t_* + \beta$$

containing an interface $t = t_* = j\tau$, where $j$ is an integer, separating material 1 below it from material 2 above it (Fig. 1). The integral of the right hand side of (33) is

$$\frac{1}{2} \int_{z_0}^{z_1} dz \int_{t_* - \beta}^{t_* + \beta} \left[ \frac{\partial}{\partial t} \left( \frac{1}{\rho} \right) \left( \frac{\partial u}{\partial t} \right)^2 + \frac{\partial k}{\partial t} \left( \frac{\partial u}{\partial z} \right)^2 \right] dt \rightarrow$$

$$\frac{1}{2} \int_{z_0}^{z_1} \left\{ \left[ \frac{1}{\rho}_{(1)} \left( \frac{\partial u}{\partial t} \right)^2 + [k]_{(1)} \left( \frac{\partial u}{\partial z} \right)^2 \right] \right\} dz, \quad (34)$$

as $\beta \to 0$, since the quantities $\rho \frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial z}$ are continuous across horizontal interfaces. Since materials 1 and 2 form a regular pairing,

$$a_{(1)} < a_{(2)} \text{ such that } \rho_{(1)} > \rho_{(2)}, \ k_{(1)} < k_{(2)},$$

due to the equality of their impedances $\gamma = \sqrt{k\rho}$, we conclude that the expression (34) is positive.
Integrating the left hand side of (33) over the horizontal strip gives
\[
\int_{z_0}^{z_1} \int_{t, -\beta}^{t, +\beta} \left( \frac{\partial}{\partial t} W_{tt} + \frac{\partial}{\partial z} W_{tz} \right) dt \, dz = \int_{z_0}^{z_1} [W_{tt}]_{t, -\beta}^{t, +\beta} \, dz + \int_{t, -\beta}^{t, +\beta} [W_{tz}]_{z_0}^{z_1} \, dt.
\]
By assuming that \([W_{tz}]_{z_0}^{z_1}\) is bounded and by passing to the limit \(\beta \to 0\), we get
\[
\left[ \int_{z_0}^{z_1} W_{tt} \, dz \right]_{t, -\beta}^{t, +\beta} = \frac{1}{2} \int_{z_0}^{z_1} \left\{ \left[ \frac{1}{\rho} \right]_{(1)}^{(2)} \left( \frac{\partial u}{\partial t} \right)^2 + [k]_{(1)}^{(2)} \left( \frac{\partial u}{\partial z} \right)^2 \right\} \, dz,
\]
from (33) and (34). In other words, the energy
\[
\int_{z_0}^{z_1} W_{tt} \, dz
\]
\textit{increases} across the horizontal interface by the amount (34).

Fig. 21. The group of characteristics in the vicinity of a limit cycle.

Consider now a group of characteristics in close vicinity of a limit cycle (Figure 21). As they approach the cycle, the horizontal distance between two neighbouring characteristics decreases from \(h\) at moment \(t = 0\) to \(h\alpha^2\) at moment \(t = \tau\), where \(\alpha = \tan \varphi_1 / \tan \varphi_2 = a_{(1)}/a_{(2)} = 0.545\) referring to the figure.

Integrate equation (33) over the domain \(ABCDEA\) in Figure 21 bounded by two horizontal segments \(AE\) and \(CD\), and by three segments \(AB, BC,\) and \(DE\) of characteristics. The horizontal segment \(AE\) is traversed along its top side \(t = +0\), whereas the segment \(CD\) is traversed along its bottom side \(t = n_1 \tau - 0\). Since the energy density flux \(W_{tz} = -k \frac{\partial u}{\partial t} \frac{\partial u}{\partial z}\) across the vertical segment \(EB\)
remains continuous because of the continuity of $k u_z$ and $u$ across vertical interfaces, and since the energy density flux across the segments $AB, BC$, and $DE$ is zero, we conclude that the energy

$$w_1 = \left. \int_A^E W_{tt} \right|_{t=0^+} dz$$

is the same as the energy

$$\int_C^D W_{tt} \bigg|_{t=n_1 \tau - 0} dz.$$

On the other hand, when we go across the segment $CD$ from its bottom side $t = n_1 \tau - 0$ occupied by material 1 to its top side $t = n_1 \tau + 0$ occupied by material 2, then the energy increases from the value $w_1$ to the value

$$w_2 = w_1 + \frac{1}{2} \int_C^D \left\{ \frac{1}{\rho(1)} \left( \frac{\partial u}{\partial t} \right)^2 + \left[ k(1)^2 \left( \frac{\partial u}{\partial z} \right)^2 \right] \right\} dz$$

$$= w_1 + \frac{a(2) - a(1)}{2a(1)} \int_C^D \left\{ \frac{1}{\rho(1)} \left( \frac{\partial u}{\partial t} \right)^2 + \gamma \left( \frac{\partial u}{\partial z} \right)^2 \right\} dz$$

$$= w_1 + \frac{a(2) - a(1)}{2a(1)} \left\{ \rho(1) \left. \left( \frac{\partial u}{\partial t} \right)^2 \right|_{t=n_1 \tau - 0} + k(1) \left. \left( \frac{\partial u}{\partial z} \right)^2 \right|_{t=n_1 \tau - 0} \right\} dz$$

$$= w_1 + \frac{a(2) - a(1)}{a(1)} w_1$$

$$= \frac{a(2)}{a(1)} w_1.$$

By a similar argument applied to the domain bounded by the contour $CGHIJD$ in Figure 21, we conclude that the energy $w_3$ on the top side of the segment $HI$ is linked to the energy $w_1$ on the top side of the segment $AE$ by the relation

$$w_3 = \left( \frac{a(2)}{a(1)} \right)^2 w_1.$$

In other words, the energy increases by the factor $\left( \frac{a(2)}{a(1)} \right)^2$ through each temporal period. Since $a(2)/a(1) > 1$, energy grows exponentially as a group of characteristics converges to a limit cycle. At each encounter with a vertical interface from material 2 to material 1, the spatial derivative of the disturbance increases, while the temporal derivative remains continuous. Similarly, the temporal derivative grows as the disturbance passes from material 1 to material 2 through a horizontal interface, while the spatial derivative remains continuous. A rectangular microstructure with materials possessing identical wave impedances may therefore accumulate energy when the characteristic pattern forms limit cycles. In other words, such a microstructure demonstrates instability of a parametric resonance type that is similar to what develops in
a swing when the energy is pumped into the system at duly chosen instants of
time. The difference is that, in our case, the instability arises for all frequen-
cies, whereas, in the case of a swing, this instability develops only for those
frequencies that exceed a certain threshold value.

When the microstructure is laminar, no accumulation of energy occurs for
low frequency waves. The reason is that, for such waves, the amount of energy
pumped into the system as the disturbance goes across an interface separating
material 1 from material 2 is balanced by the energy taken away from the
system at the subsequent interface where material 2 is followed by material 1.
In this respect, the system resembles a simple harmonic oscillator in which the
energy is periodically transformed from potential to kinetic, and vice versa,
but the total energy of the system is preserved. For waves of higher frequency,
however, this energy balance in a laminate may not occur and the system may
then become unstable just as in the case of a swing.

Returning to the rectangular microstructure, it is interesting to note, as men-
tioned at the end of the previous section, that the solution tends to become
continuous as \( \delta, \tau \to 0 \), whereas the energy needed to maintain propagation of
low frequency waves through a microstructure that gives rise to limit cycles
may become infinite. Of course, in such circumstances, there is no homoge-
neization in its standard version applicable to laminates.

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References

Isotropic Dielectrics Distributed in Space-Time and the Conservation Law for
Wave Impedance in One-Dimensional Wave Propagation,” Proc. Royal Society

[2] Eichler, H. J., “Introduction to the Special Issue on Dynamic Gratings and
(1986).

Devices and Applications,” IEEE Transactions on Magnetics 32, 4118-4223
(1996).


