1. Give a **formula** and then **evaluate** the following expressions. Your numerical answers should be rounded to four decimal places. *(15 points)*

a. \( \ddot{s}_{8|} \) assuming \( i = 10\% \).

\[
\frac{(1+i)^n - 1}{d} = \frac{(1.1)^8 - 1}{0.1110} = 12.5795
\]

b. \( a_{20|} \) assuming \( i = 9\% \).

\[
\frac{1 - v^n}{i} = \frac{1 - (\frac{1}{1.09})^{20}}{0.09} = 9.1285
\]

c. \( s_{100|} \) assuming \( i = 8\% \).

\[
\frac{(1+i)^n - 1}{i} = \frac{(1.08)^{100} - 1}{0.08} = 27,484.5157
\]

d. \( \ddot{a}_{\infty} \) assuming \( i = 7\% \).

\[
\frac{1}{d} = \frac{1}{(0.07/1.07)} = 15.2857
\]

e. \( a_{7|} \) assuming \( i = 6\% \).

\[
V^n a_{\bar{n}|} = V^0 \left( \frac{1-v^7}{i} \right) = 3.1172
\]
2. A perpetuity-due which pays $1,000 per year and costs $12,000 is equivalent to a perpetuity of $X per year which has its first payment seven years from today. What is $X$, to the nearest penny? (10 points)

\[ 1000 \ddot{a}_{\overline{\infty}} = X \times v^n \ddot{a}_{\overline{\infty}} \]

\[ 1000 \ddot{a}_{\overline{\infty}} = 12000 \implies d = \frac{1}{12}, \ c = \frac{1}{11}, \ v = \frac{11}{12} \]

\[ X = \frac{12000}{v^6 \ddot{a}_{\overline{\infty}}} = \frac{12000}{\left(\frac{11}{12}\right)^6 \left(\frac{1}{11}\right)} \]

\[ X = \$1838.74 \]
3. You contribute $100 to a fund at times $t = 0, 1, 2, \ldots 38, 39$ (so, forty contributions in all).

You then withdraw $200 from the fund at times $t = 41, 42, 43, \ldots 59, 60$ (so, twenty withdrawals in all).

Given an effective annual interest rate of 10% in years 1 to 40, and an effective annual interest rate of 8% thereafter, how much is in the fund at $t=60$, immediately following the last $200$ withdrawal? Please express your answer to the nearest penny. (10 points)

\[ \text{Fund} = (100 \ddot{S}_{40|0.10})(1.08)^{20} + (-200 \ddot{S}_{20|0.08}) \]

\[ \text{Fund at } t=40 \quad \text{Accumulate from } t=40 \text{ to } t=60 \quad \text{Accumulate the withdrawals} \]

\[ = 226,919.54 - 9152.39 \]

\[ = \$217,767.15 \]
4. Using standard actuarial notation, give \textbf{two different expressions} for the value at \( t=6 \) of the series of payments shown in the following table.

[Note: As an example of "standard actuarial notation", the value at \( t=0 \) of $100 payments occurring at \( t = 1, 2, 3, 4, \) and 5 could be expressed as either \( "100a_{\overline{5}|i}" \) or \( "100v^6\ddot{s}_{\overline{5}|i}" \) – many other answers are possible!]

<table>
<thead>
<tr>
<th>( t )</th>
<th>Payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>200</td>
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<tr>
<td>2</td>
<td>200</td>
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<td>3</td>
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<td>4</td>
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<td>7</td>
<td>100</td>
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<tr>
<td>8</td>
<td>100</td>
</tr>
<tr>
<td>9</td>
<td>100</td>
</tr>
<tr>
<td>10</td>
<td>100</td>
</tr>
</tbody>
</table>

\begin{itemize}
\item \textbf{a. First expression} (8 points)
\[ 200 \, s_{\overline{6}|i} + 100 \, a_{\overline{4}|i} \]
\item \textbf{b. Second expression} (7 points)
\[ (100 \, a_{\overline{10}|i} + 100 \, a_{\overline{8}|i})(1+i)^6 \]
\end{itemize}
5. At time \( t=3 \), the current value of \( \mathbf{A} + \mathbf{B} \) is equivalent to the current value of \( \mathbf{C} + \mathbf{D} \), where:

- \( \mathbf{A} \) is a fifteen year annuity of $X$ per year, first payment at time \( t=1 \)
- \( \mathbf{B} \) is $5000 payable at time \( t=5 \)
- \( \mathbf{C} \) is an annual perpetuity with first payment of $600 at time \( t=7 \)
- \( \mathbf{D} \) is $10,000 payable at time \( t=10 \)

If \( i=6\% \), what is \( X \), to the nearest penny? \textit{Note:} For best partial credit (in case you make a mistake someplace), it would be good if you separately give expressions for \( \mathbf{A} \), \( \mathbf{B} \), \( \mathbf{C} \), and \( \mathbf{D} \) – up to you, of course… \( (10 \text{ points}) \)

\[
\begin{align*}
\mathbf{A} &= X \cdot a_{15}^{0.06} \\
\mathbf{B} &= 5000 \cdot v^5 = 3736.29 \\
\mathbf{C} &= 600 \cdot v^6 \cdot a_{\infty}^{0.06} = 7049.61 \\
\mathbf{D} &= 10000 \cdot v^{10} = 5583.95 \\
\mathbf{A} + \mathbf{B} = \mathbf{C} + \mathbf{D} \implies X &= \frac{\mathbf{C} + \mathbf{D} - \mathbf{B}}{a_{15}^{0.06}} \\
X &= \frac{7049.61 + 5583.95 - 3736.29}{9.712249} \\
X &= \frac{15001.46}{9.712249} \\
X &= 1538.87 \\
X &= \$1916.09
\end{align*}
\]
6. You borrow $300,000 to buy a new home. You will repay the loan over thirty years, in equal monthly payments beginning one month from today (360 monthly payments in all). The nominal rate of interest, convertible monthly, is 12%.

a. How much is the monthly payment? Please give you answer to the nearest penny. \((7 \text{ points})\)

\[
\text{Payment} = \frac{300,000}{\alpha_{360|0.01}} = 3,085.84
\]

b. How much interest will you pay over the life of the loan? Please express your answer in whole dollars. \((3 \text{ points})\)

\[
\begin{align*}
\text{Total payments over life of loan} &= (360)(3085.84) \\
\text{Total principal} &= 300,000
\end{align*}
\]

\[
\Delta = \text{total interest} = \#810,902
\]
7. Al, Bob, and Charlie are dividing a perpetuity-due into three equal pieces. Al gets $X\%$ of every payment, Bob gets $(1-X\%)$ of the first 12 payments, and Charlie gets $(1-X\%)$ of all payments thereafter.

What is the effective interest rate? Note I am looking for an actual percentage, not an expression in terms of $X\%$. 

\[ A = B = C = \frac{1}{3} \text{ of total} \]

\[ \frac{1}{3} \ddot{a}_{\infty} = \frac{2}{3} \ddot{a}_{12} \]

\[ \frac{1}{d} = 2 \left( \frac{1-v^{12}}{d} \right) \]

\[ v^{12} = \frac{1}{2} \]

\[ v = 12\sqrt{\frac{1}{2}} = 0.943874312 \]

\[ i = \frac{1}{v} - 1 = 5.95\% \]
BONUS QUESTION  (up to 10 points; quiz score cannot exceed 100%)

What is the present value of a perpetuity-immediate which pays at a decreasing rate, such that the first payment is $100, and each subsequent payment is 99% of the previous payment? Express your answer to the nearest penny. Assume $i=8\%$.

\[
\begin{align*}
S &= \frac{100}{1.08} + \frac{(100)(0.99)}{1.08^2} + \frac{(100)(0.99)^2}{1.08^3} + \cdots \\
\frac{0.99}{1.08} S &= \frac{(100)(0.99)}{1.08^2} + \frac{(100)(0.99)^2}{1.08^3} + \cdots \\
S &= \frac{100}{1.08} - \frac{0.99}{1.08} \\
S &= \frac{100}{1.08 - 0.99}
\end{align*}
\]

\[S = \$ 1111.11\]

Note: In general, for a perpetuity increasing \( j \) per year, \( PV = \frac{1}{i+j} \)

***END OF QUIZ***