1. (6 points) A password for the website www.youcantrustus.ru has 8 symbols from \(A \cup D\), where \(A = \{a, b, c, \ldots, x, y, z\}\) is the set of characters and \(D = \{0, 1, 2, 3, \ldots, 8, 9\}\) is the set of digits. For the password to be acceptable, it must either start with three digits, or end with four characters, or consist only of symbols from \(\{1, 2, 3, a, b, c\}\).

How many legal passwords are there?

\[\textsf{We use inclusion exclusion. Let } X = \text{ set of passwords with the first three symbols in } D, Y = \text{ set of passwords whose last four symbols are in } A, \text{ and } Z = \text{ passwords with all symbols from } \{1, 2, 3, a, b, c\}.\]

The acceptable passwords must satisfy at least one of the conditions so we want to compute \(|X \cup Y \cup Z|\) and so use inclusion exclusion so that each computation only uses the multiplicative principle.

First we compute: \(|X| = 10^3 \cdot 36^5\), \(|Y| = 36^4 \cdot 26^4\), and \(|Z| = 6^8\).

Next we compute \(|X \cap Y| = 10^3 \cdot 36 \cdot 26^4\), \(|X \cap Z| = 3^3 \cdot 6^5\), and \(|Y \cap Z| = 6^4 \cdot 3^4\), and \(|X \cap Y \cap Z| = 3\cdot 6 \cdot 3^4\).

Now, using inclusion exclusion we have

\[
|X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |X \cap Z| - |Y \cap Z| + |X \cap Y \cap Z|
\]

\[
= (10^3 \cdot 36^5) + (36^4 \cdot 26^4) + (6^8)
\]

\[
- (10^3 \cdot 36 \cdot 26^4) - (3^3 \cdot 6^5) - (6^4 \cdot 3^4)
\]

\[
+ (3^3 \cdot 6 \cdot 3^4)
\]

2. (4 points) Label each of the following TRUE or FALSE.

a) \(\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\) \(|\mathcal{P}\{1, 2, 3, 4, 5, 6\}\rangle < |\mathcal{P}_4\{1, 2, 3, 4, 5, 6, 7, 8\}\rangle|\)

\(\textbf{TRUE: } |\mathcal{P}\{1, 2, 3, 4, 5, 6\}\rangle = 2^6 = 64. \ |\mathcal{P}_4\{1, 2, 3, 4, 5, 6, 7, 8\}\rangle| = \binom{8}{4} = 70.\)

b) \(\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\) \(|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|\)

\(\textbf{TRUE: } \text{Finite product of countable sets is countable.}\)

c) \(\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\) \(|\mathbb{N} \times \mathbb{N}| = |\mathbb{Q}|\)

\(\textbf{TRUE: } \text{Both } \mathbb{N} \times \mathbb{N} \text{ and } \mathbb{Q} \text{ are countable.}\)

d) \(\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\) \(|\mathcal{P}(\mathbb{N})| > |\mathbb{Q} \times \mathbb{Q}|\)

\(\textbf{TRUE: } \mathcal{P}(\mathbb{N}) \text{ is uncountable. } \mathbb{Q} \times \mathbb{Q} \text{ is countable.}\)