1. Find a regular expression for the language of the grammar:

\[
S \rightarrow aA | bA | \lambda \\
A \rightarrow aA | bS
\]

There are many ways to analysis this. Here are two.

A has one recursive and one non-recursive rule. Using only the recursive rule you get
\[A \rightarrow a^nA\], and so in general we get \[A \rightarrow a^n bS\], so we can replace each \[A\] is the rules for \[S\] with \[a^*bS\]:

\[
S \rightarrow aS | ba^*bS | \lambda
\]

Now \[S\] has two recursive rules, so we get \[(a \cup ba^*b)^*\]. which describes all strings with an even number of b’s.

Or, you can notice that each variable \[S\] and \[A\] is recursive in \(a\), but they switch when the prefix is augmented by \(b\). So you can assign the meanings

\[S: The prefix has an even number of b’s.\]

\[A: The prefix has an odd number of b’s.\]

The truth of these statements is preserved under all the rules of the grammar.

2. Give a regular grammar for the set of strings in \(\{a, b\}\) such that the number of \(a\)'s is divisible by 3.

Again, there are many ways to solve this.

Let’s chose variables \(S\), \(A\) and \(B\) and give them the meanings:

\[S: The prefix \(p\) has \(n_a(p)\) divisible by 3.\]

\[A: The prefix \(p\) has \(n_a(p)\) with remainder 1 when divided 3.\]

\[B: The prefix \(p\) has \(n_a(p)\) with remainder 2 when divided 3.\]

This leads us to construct the grammar with the rules:

\[
S \rightarrow aA | bS | \lambda \\
A \rightarrow aB | bA \\
A \rightarrow aS | bB
\]

Although it wasn’t asked, we can easily prove the result is correct. The statements is true for the Start symbol with the empty prefix, and the truth of each statement preserved by each rule application, so it is satisfied by every sentential form. Since the only sentential forms in \(\{a, b\}^*\) occur after \(S \rightarrow \lambda\), every word in the language has the number of \(a\)'s divisible by 3. This shows \(L(G) \subseteq L\).

To show \(L(G) \subseteq L\), we note that since for each variable we may augment the prefix by either \(a\) or \(b\), all elements of \(\{a, b\}^*\) are prefixes of a sentential form, with those with \(n_a\) divisible by 3 prefixes of \(S\), which is nullable, so \(L(G) \subseteq L\).
3. Let

\[
\begin{align*}
S & \rightarrow aAa \mid B \\
B & \rightarrow bbBdd \mid C \\
C & \rightarrow bd
\end{align*}
\]

Give a derivation sequence for \( a^j b^{2k+1} d^{2k+1} a^{2j} \).

\[
S \xrightarrow{\ast} a^j S a^{2j} \xrightarrow{\ast} a^j B a^{2j} \xrightarrow{k} a^j b^{2k} B d^{2k} a^{2j} \xrightarrow{\ast} a^j b^{2k} C d^{2k} a^{2j} \xrightarrow{\ast} a^j b^{2k} b d d^{2k} a^{2j} = a^j b^{2k+1} d^{2k+1} a^{2j}
\]