Lectures 21 and 22

Because of Patriot’s Day and PPD, these comprised the whole week.
We discussed recursion.
We gave recursive constructions of the factorials and the binomial coefficients.
Solved the Fibonacci recursion
\[ f_{n+1} = f_n + f_{n-1} \quad f_0 = 0, f_1 = 1 \]
We showed the binomial theorem
\[ (a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \]
We showed how to solve homogeneous linear recursions:
\[ A_0 a_n + A_1 a_{n+1} + A_2 a_{n+2} + \ldots + A_k a_{n+k} = 0 \]
with initial conditions \( a_0, a_1, \ldots, a_k \), by looking for exponential solutions \( a_n = C r^n \), which give the characteristic equation
\[ A_0 + A_1 r + A_2 r^2 + \ldots + A_k r^k = 0 \]
for \( r \), with \( k \) solutions, which can be combined satisfy the initial conditions.
We used this to solve \( a_{n+1} = 3a_n - 2a_{n-1} \); \( a_0 = 0, a_1 = 1 \).
We described the Towers of Hanoi problem. We showed that for 3 spindles and \( n \) disks there are \( 3^n \) legal positions by the multiplicative principle.
We showed there is a recursive solution to the problem of how many moves \( m_n \) to solve the \( n \)-disk problem:
\[ m_n = 2m_{n-1} + 1 \quad m_1 = 1 \]
and solved the linear recursion to compute that the solution has \( 2^n - 1 \) moves.

Exercises for Lectures 21 and 22

1. Let \( T_n = \sum_{k=0}^{n} k^2 \). (Our cannonball numbers if you remember.)
   Write a recursive specification for this sequence. Is it linear? Do not forget the initial conditions.
2. Let \( s_k = (2k)!/(2^k k!) \).
   Write a recursive specification for this sequence. Do not forget the initial conditions.
3. Suppose \( a_0 = 3 \) and for \( n \geq 1 \) we have \( a_{n+1} = 6a_n \).
   Find a formula for \( a_n \).
4. Suppose $a_0 = 3$ and for $n \geq 1$ we have $a_{n+1} = 6a_n - 6$.
   Find a formula for $a_n$.

5. Suppose $a_0 = 1$ and $a_1 = 2$, and for $n \geq 1$ we have $a_{n+1} = 6a_n - 8a_{n-1}$.
   Find a formula for $a_n$.

6. Suppose $a_0 = 1$ and $a_1 = 2$, and for $n \geq 1$ we have $a_{n+1} = 6a_n - 8a_{n-1} + 2$.
   Find a formula for $a_n$.

7. Suppose $a_0 = 5$ and $a_1 = 10$, and for $n \geq 1$ we have $a_{n+1} = 7a_n - 12a_{n-1}$.
   Find a formula for $a_n$.
   What is $\lim_{n \to \infty} a_n$?

8. According to the Binomial Theorem what is
   \[
   \binom{7}{0} + \binom{7}{1}2 + \binom{7}{2}4 + \binom{7}{3}8 + \binom{7}{4}16 + \binom{7}{5}32 + \binom{7}{6}64 + \binom{7}{7}128
   \]

9. According to the Binomial Theorem what is
   \[
   \binom{7}{0} - \binom{7}{1}2 + \binom{7}{2}4 - \binom{7}{3}8 + \binom{7}{4}16 - \binom{7}{5}32 + \binom{7}{6}64 - \binom{7}{7}128
   \]

10. According to the Binomial Theorem what is
    \[
    \binom{7}{0} - \binom{7}{1} + \binom{7}{2} - \binom{7}{3} + \binom{7}{4} - \binom{7}{5} + \binom{7}{6} - \binom{7}{7}
    \]

11. According to the Binomial Theorem what is
    \[
    \binom{7}{0} - \binom{7}{1} \frac{1}{2} + \binom{7}{2} \frac{1}{4} - \binom{7}{3} \frac{1}{8} + \binom{7}{4} \frac{1}{16} - \binom{7}{5} \frac{1}{32} + \binom{7}{6} \frac{1}{64} - \binom{7}{7} \frac{1}{128}
    \]