Exercises for Lectures 5 and 6

Cardinality of Finite Sets:
Lectures 3 and 4 we defined the cartesian product of two sets and discussed
the cardinality of finite sets.
Among other things that
$$|\mathcal{P}(A)| = 2^{|A|} \quad |A \times B| = |A| \cdot |B| \quad |\mathcal{P}_k(A)| = \binom{|A|}{k} = \frac{|A|!}{k!(|A| - k)!}$$

We discussed encoding the subsets of $A$ by bit vectors, and the connection
to binary numbers.
We discussed the the inclusion/exclusion principle.
$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D|$$
$$-|A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D|$$
$$+|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|$$
$$-|A \cap B \cap C \cap D|$$

We showed the relation $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ and used it to construct
Pascal’s Triangle.

1. Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$. Construct the sets $A \times B$, $\mathcal{P}(A) \times (B \times A)$, and $\mathcal{P}_2(A) \times \mathcal{P}_2(B)$.
2. Let $X = \{1, 2, \ldots, 20\}$ and $B = \{1, 2, \ldots, 30\}$. Compute $|X \times Y|$, $|\mathcal{P}(X) \times (Y \times X)|$, and $|\mathcal{P}_2(X) \times \mathcal{P}_2(Y)|$.
3. Let $X$, $Y$, and $Z$ be finite sets. Show that $|X \times (Y \times Z)| = |Z \times (Y \times X)|$.
4. Let $A$ and $B$ be finite sets, and suppose the $|A \cup B| = 100$ and $|A \cap B| = 50$.
   What is the largest value that $|A|$ can have? What can you say about $B$ in this case?
   What is the smallest value that $|A|$ can have? What can you say about $B$ in this case?
5. Suppose $|\mathcal{P}(A)| = |A \times A|$. What can you say about $A$.
6. Suppose $|\mathcal{P}(A)| = 4|A|$. What can you say about $A$.
7. Give an example of a set $X$ with a subset $Y \subseteq X$ such that $|\mathcal{P}(Y) = X|$.
8. How many 10 digit numbers (5 = 0000000005 is also a 10 digit number) are either divisible by 1000 or divisible by 2?
9. How many 8 letter words either start with three vowels (a, e, i, o, u) or end with two vowels?

10. How many 9 letter words either start with 5 identical characters, like xxxxxfyry, end with 5 identical characters, like rorooooo, or repeat in groups of three, like blablabla.

11. How many 7 letter words either start with x, end with x, or read the same forwards or backwards.

12. How many 6 letter words either start with two identical characters, end with two e’s, read the same forwards or backwards, or repeat after 3 letters.