Introduction: Wave Optics
January 2019
Wave optics: light waves

Wave equation

An optical wave -- monochromatic -- can be described mathematically by the complex wavefunction

\[ U(x, y, z, t) = a(x, y, z) \cdot \exp[j\phi(x, y, z)] \cdot \exp[j2\pi\nu t] \]  \hspace{1cm} (1)

where

- \(x, y, z\) are the components of the position vector \(r\)
- \(t\) is time
- \(\phi(x, y, z)\) is the optical phase
- \(a(x, y, z)\) is the amplitude
- \(\nu\) is the frequency [Hz]
  \(\omega = 2\pi\nu = \) angular frequency [rad/sec]
- \(j\) is the complex quantity \(\sqrt{-1}\)
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Wave equation

Equation (1) can be written as

\[ U(x, y, z, t) = U(x, y, z) \cdot \exp[j2\pi \nu t] \]  

(2)

where the time-independent term,

\[ U(x, y, z) = a(x, y, z) \cdot \exp[j\phi(x, y, z)] \]  

(3)

is the complex amplitude of the optical wave \( U(x, y, z, t) \)
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Wave equation

Function $U(x, y, z, t)$ must satisfy the wave equation (in order to represent a valid wave function), therefore,

$$\nabla^2 U - \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} = 0$$

(4)

where

$$c = \frac{c_o}{n} \quad \text{for} \quad n \geq 1$$

(5)

in which $c_o$ is the speed of light in free-space and the wave propagates in a medium with index of refraction $n$. 
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Wave equation

By substituting Eq. 2 into the wave equation, Eq. 4, the following equation is obtained -- exercise in class/homework

\[(\nabla^2 + k^2)U(x, y, z) = 0\]  \hspace{1cm} (6)

which is called the Helmholtz equation, where

\[k = \frac{2\pi v}{c} = \frac{\omega}{c} = \frac{2\pi}{\lambda}\]  \hspace{1cm} (7)

in the wave number, and \(\lambda\) is the spatial wavelength.

Note that:

\[\lambda = \frac{c}{v}\]  \hspace{1cm} (8)
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Elementary waves

The two canonical solutions of the Helmholtz equation in a homogenous medium are: (1) the plane wave, and (2) the spherical wave.

(1) The plane wave

The plane wave has the complex amplitude:

\[ U(x, y, z) = A \exp(-j \mathbf{k} \cdot \mathbf{r}) \]  

\[ = A \exp[-j(k_x \cdot x + k_y \cdot y + k_z \cdot z)] \]

where

- \( A \) is the amplitude, or complex envelope
- \( \mathbf{k} = k_x \hat{i} + k_y \hat{j} + k_z \hat{k} = (k_x, k_y, k_z) \) is the propagation direction vector
- \( \mathbf{r} = x \hat{i} + y \hat{j} + z \hat{k} = (x, y, z) \) is the position vector
- \( j \) is the complex quantity \( \sqrt{-1} \)
Elementary waves

Plane waves

For Eq. 9 to satisfy the Helmholtz equation, Eq. 6, it is necessary that

\[ k_x^2 + k_y^2 + k_z^2 = k^2 \]  

(10.1)

that is, the magnitude of the propagation direction vector, \( k \), is equal to the wave number, \( k \),

\[ |k| = k \]  

(10.2)
Elementary waves

Plane waves

Since the phase, or \( \arg[U(x, y, z)] = \arg[A] - k \cdot r \), the wavefronts are

\[
k_x \cdot x + k_y \cdot y + k_z \cdot z = 2\pi q + \arg[A]
\]

for \( q = \text{integer} \)

Equation 11 describes the family of parallel planes that is perpendicular to the propagation direction vector, \( k \). These planes are called: wavefronts.

Planes are separated by the distance

\[
\lambda = \frac{2\pi}{k}
\]
Elementary waves

Plane waves

If the $z$-axis is taken in the direction of the propagation vector, $k$, then

$$U(z) = A \exp(-j k z)$$  \hspace{1cm} (13)

using Eqs 13 and 2,

$$U(z, t) = A(z) \cdot \exp[-j k z] \cdot \exp[j 2\pi \nu t]$$

$$= A(z) \cdot \exp[j (2\pi \nu t - k z)]$$

$$= |A| \cdot \exp\{j (2\pi \nu t - k z + \arg[A])\}$$  \hspace{1cm} (15)

and by separating the real component of Eq. 15, it is obtained

$$u(z, t) = \text{Re}\{U(z, y)\} = |A| \cdot \cos(2\pi \nu t - k z + \arg[A])$$

$$= |A| \cdot \cos\{2\pi \nu(t - \frac{z}{C}) + \arg[A]\}$$  \hspace{1cm} (16)
Elementary waves

Plane waves

A plane wave traveling in the $z$-direction is a periodic function of $z$ with spatial period $\lambda$ and a periodic function of $t$ with temporal period $1/\nu$. 
Elementary waves

Observations

- **Optical phase**, obtained from Eq. 16,

\[
\text{arg}[\text{Re}\{ U(z,t) \}] = 2\pi \nu(t - \frac{z}{c}) + \text{arg}[ A]
\]  

(17)

varies as a function of time and position

- **Optical intensity** is determined as

\[
I = |U|^2 = U \cdot U^* \]

(18)

where

\[ U^* \text{ is the complex conjugate of } U \]
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Elementary waves

(2) The spherical wave

The spherical wave is another canonical solution of the Helmholtz equation. Its complex amplitude is

\[ U(r) = \frac{A}{r} \exp(-j k r) \]  

(19)

where \( r \) is the distance from the propagation origin.

\[ k = \frac{2\pi}{\lambda} \]

is the wave number, and

\[ I = U \cdot U^* = \frac{|A|^2}{r^2} \]

(proportional to the square of the distance from the origin)
Elementary waves

Spherical waves

Taking $\arg[A] = 0$, for simplicity,

$$k \cdot r = k \sqrt{x^2 + y^2 + z^2} = 2\pi q + \arg[A]$$  \hspace{1cm} (20)

for $q = \text{integer}$

Equation 20 describes the family of concentric spheres: spherical wavefronts.

Spheres are separated by the distance

$$\lambda = \frac{2\pi}{k}$$  \hspace{1cm} (21)

Cross-section of the wavefronts of a spherical wave
The rays of ray optics are orthogonal to the wavefronts of wave optics. Note the effect of a lens on rays and wavefronts.
Optical interference

Interferometers

Mach-Zender

![Mach-Zender Interferometer Diagram]
Optical interference

Interferometers

Michelson

Beamsplitter
Optical interference

Interferometers

Sagnac

Beam splitter

$U_0$ $U_1$ $U_2$ $U$
Optical interference
Holographic interferometry

Recording

Reconstruction
Interference equation

Consider the superposition of two monochromatic plane waves $U_1$ and $U_2$ from the same light source

$$U = U_1 + U_2$$ (22)

The corresponding intensity is,

$$I = |U|^2 = |U_1 + U_2|^2 = |U_1|^2 + |U_2|^2 + U_1 U_2^* + U_1^* U_2$$ (23)

if $U_1 = A_1 \exp(-j k_1 \cdot r) = A_1 \exp(-j \phi_1)$,

$U_2 = A_2 \exp(-j k_2 \cdot r) = A_2 \exp(-j \phi_2)$

The observed intensity, measured, is

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos(\phi_2 - \phi_1)$$ (24)
Interference equation, cont’d

\[ I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos(\phi_2 - \phi_1) \]  

Defining: \[ I_B = I_1 + I_2 = \text{Background intensity} \]
\[ I_M = 2\sqrt{I_1 I_2} = \text{Modulation intensity} \]

Intensity becomes:

\[ I = I_B + I_M \cos(\Delta\phi) \]