1. Suppose that $S(x)$ is the natural cubic spline passing through the points $(x_i, f(x_i))$, $i = 0, 1, 2, \ldots, n$ where $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$.

   a) Prove that
   
   $$\int_a^b [S''(x)]^2 \, dx \leq \int_a^b [f''(x)]^2 \, dx.$$

   b) What if $S(x)$ is a clamped spline? Is it still true that
   
   $$\int_a^b [S''(x)]^2 \, dx \leq \int_a^b [f''(x)]^2 \, dx?$$

2. Find the corresponding cubic splines $S(x)$ satisfying

   $$S_0(1) = 2, \quad S_0(2) = S_1(2) = 1, \quad S_1(3) = 1$$

   $$S_0'(2) = S_1'(2), \quad S_0''(2) = S_1''(2),$$

   with the following boundary conditions

   $$S_0''(1) = S_1''(3) = 0 \text{ (natural splines)}$$

   $$S_0'(1) = 1, \quad S_1'(3) = 2 \text{ (clamped splines)}$$

   $$S_0'''(2) = S_1'''(2), \text{ (not-a-knot condition)}$$

3. Plot the errors of three cubic splines, (clamped, natural, not-a-knot) using 5, 20 equispaced points at $x_j = -1 + 2j/500$, $0 \leq j \leq 500$ interpolating the Runge function $f(x) = \frac{1}{1+25x^2}$ over $[-1, 1]$. Create two graphs: the first one contains only errors for splines using 5 equispaced points and the other contains errors for splines using 20 equispaced points. Use the Matlab command ‘semilogy’ in your plot and also ‘legend’ to indicate different splines in each graph.

4. (Building quadratic splines). Given a smooth function $f$ on $[a, b]$ and nodes

   $$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

   Find a quadratic spline $S(x)$ (with $S'(x_0) = 0$) interpolating $f$ using the following steps:

   - formulate $S(x) =: S_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2$ for $x \in [x_i, x_{i+1}]$, $i = 0, 1, 2 \cdots, n-1$.
   - write down the resulting linear system in terms of $b_i$.
   - solve the linear system and write down explicit forms of $S_i(x)$.
1. \[ \text{Proof.} \text{ Define } g(x) = f(x) - S(x). \text{ Then since } S(x) \text{ interpolates } f(x) \text{ at } x_i, \text{ then } g(x_i) = 0, \text{ } i = 0, 1, 2, \cdots n. \]

\[
\int_a^b [f''(x)]^2 \, dx = \int_a^b [S''(x) + g''(x)]^2 \, dx
\]

\[
= \int_a^b [S''(x)]^2 \, dx + \int_a^b [g(x)]^2 \, dx + 2 \int_a^b S''(x) g''(x) \, dx
\]

\[
= \int_a^b [S''(x)]^2 \, dx + \int_a^b [g(x)]^2 \, dx + 2 \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} g''(x) S''(x) \, dx
\]

\[
= \int_a^b [S''(x)]^2 \, dx + \int_a^b [g(x)]^2 \, dx + 2 \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} g'(x) S'''(x) |_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} g'(x) S''(x) \, dx
\]

By the definition of splines, we have that \( S''(x) \) is piecewise constants and \( g(x_i) = 0 \). Then

\[
\int_a^b [f''(x)]^2 \, dx = \int_a^b [S''(x)]^2 \, dx + \int_a^b [g(x)]^2 \, dx
\]

\[
+ S''(x_n) g'(x_n) + S''(x_0) g'(x_0) - 2 \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} g'(x) S'''(x) \, dx
\]

\[
= \int_a^b [S''(x)]^2 \, dx + \int_a^b [g(x)]^2 \, dx
\]

\[
+ S''(x_n) g'(x_n) - S''(x_0) g'(x_0) - 2 \sum_{i=0}^{n-1} S'''(x_n) |_{x_i}^{x_{i+1}} g'(x) \, dx
\]

\[
= \int_a^b [S''(x)]^2 \, dx + \int_a^b [g(x)]^2 \, dx + S''(x_n) g'(x_n) - S''(x_0) g'(x_0).
\]

For natural splines, \( S''(x_0) = S''(x_n) = 0 \) while for clamped splines, \( g'(x_n) = g'(x_0) = 0 \). Then we have \( \int_a^b [f''(x)]^2 \, dx = \int_a^b [S''(x)]^2 \, dx + \int_a^b [g(x)]^2 \, dx \geq \int_a^b [S''(x)]^2 \, dx. \)

\[ \square \]

2. \textbf{Solution.} We can use the linear system to solve the problem. Or you want to write \( S_0(x) = a_0 + b_0(x-x_0) + c_0(x-x_0)^2 + d_0(x-x_0)^3 \) and \( S_1(x) = a_1 + b_1(x-x_0) + c_1(x-x_0)^2 + d_1(x-x_0)^3 \) and apply the given conditions to obtain the coefficients.

For natural splines,

\[
S_0(x) = 2 - \frac{5}{4}(x-1) + \frac{1}{4}(x-1)^3, \quad 1 \leq x \leq 2; \quad S_1(x) = 1 - \frac{1}{2}(x-2) + \frac{3}{4}(x-2)^2 - \frac{1}{4}(x-2)^3, \quad 2 \leq x \leq 3.
\]

For clamped splines,

\[
S_0(x) = 2 + (x-1) - \frac{7}{2}(x-1)^2 + \frac{3}{2}x^3, \quad 1 \leq x \leq 2; \quad S_1(x) = 1 - \frac{3}{2}(x-2) + (x-2)^2 + \frac{1}{2}(x-2)^3, \quad 2 \leq x \leq 3.
\]
With the given not-a-knot condition, we can not really determine the spline and actually we have a cubic polynomial over \([x_0, x_2]\) where \(x_0 = 1\), \(x_1 = 2\) and \(x_2 = 3\).

\[
S(x) = 1 + b(x - 2) + c(x - 2)^2 + d(x - 2)^3.
\]

Using \(S(1) = 3\) and \(S(1) = 2\), we have \(c = 1/2\) and \(b + d = -1/2\). The solution is

\[
S(x) = 1 + b(x - 2) + \frac{1}{2}(x - 2)^2 + (-\frac{1}{2} - b)(x - 2)^3.
\]

3. In Matlab, the cubic splines are implemented with ‘csape’.

```matlab
m=5; % m=20;
x= linspace(-1,1,m);
y= 1./(1+25*x.^2);
xx = -1:2/500:1;
yy= 1./(1+25*xx.^2);
natural_cubic_spline = csape (x,y,'variational'); % natural splines
clamped_cubic_spline = csape (x,[-50/676 y 50/676],[1 1]); % clamped
% Matlab uses numerical derivatives which may not be good as explained in class.
% You will observe different behavior at -1 and +1 using these two lines above.
nak_cubic_spline = csape (x,y,'not-a-knot'); % not-a-knot boundary

natural_cubic_spline_val = fnval(natural_cubic_spline,xx); % ppval can be used too.
clamped_cubic_spline_val = fnval(clamped_cubic_spline ,xx);
nak_cubic_spline_val = fnval( nak_cubic_spline, xx);

% Checking errors
natural_cubic_spline_err = (natural_cubic_spline_val-yy);
clamped_cubic_spline_err = (clamped_cubic_spline_val-yy);
nak_cubic_spline_err = ( nak_cubic_spline_val-yy);

% Figures
figure(m)
semilogy(xx, abs(natural_cubic_spline_err), 'r-*', 'Linewidth', 2, 'Markersize', 6);
hold on
semilogy(xx, abs(clamped_cubic_spline_err), 'b-o', 'Linewidth', 2,'Markersize', 6);
```
semilogy(xx, abs(nak_cubic_spline_err), 'c-d','Linewidth', 2,'Markersize', 6); hold off

set(gca, 'Fontsize', 16) title ([ ' ',num2str(m),' equispaced points in [-1,1]' ]); xlabel('x') ylabel('Absolute error') h_legend=legend('natural cubic spline', 'clamped cubic spline', 'Not-a-knot cubic spline',0); set(h_legend, 'Fontsize', 16);

saveas(m,[ 'runge_splines_',num2str(m),'.eps'],'epsc2');

% using plot instead of semilogy figure(m+1)

plot(xx, natural_cubic_spline_err, 'r-*', 'Linewidth', 2, 'Markersize', 6); hold on

plot(xx, clamped_cubic_spline_err, 'b-o', 'Linewidth', 2,'Markersize', 6);

plot(xx, nak_cubic_spline_err, 'c-d','Linewidth', 2,'Markersize', 6);

hold off

set(gca, 'Fontsize', 16) title ([ ' ',num2str(m),' equispaced points in [-1,1]' ]); xlabel('x') ylabel('Error') h_legend=legend('natural cubic spline', 'clamped cubic spline', 'Not-a-knot cubic spline',0);

set(h_legend, 'Fontsize', 16);

4. Define $S(x) = S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2$ for $x \in [x_i, x_{i+1}]$, $i = 0,1,2\cdots,n-1$. Define that $h_i = x_{i+1} - x_i$, $i = 0,1,2\cdots,n-1$. Then applying the matching conditions gives

$$
a_i = f(x_i), \quad i = 0,1,2\cdots,n-1,
$$

$$
a_i + b_i h_i + c_i h_i^2 = f(x_{i+1}), \quad i = 0,1,2\cdots,n-1,
$$

$$
b_{i+1} = b_i + 2c_i h_i, \quad i = 0,1,2\cdots,n-1.
$$

where we define $b_n = b_{n-1} + 2c_{n-1} h_{n-1}$ We then have

$$
2b_i h_i + (b_{i+1} - b_i) h_i = 2[f(x_{i+1}) - f(x_i)], \quad i = 0,1,2\cdots,n-1.
$$
and further we have a tridiagonal linear system

\[ b_{i+1} = \frac{2[f(x_{i+1}) - f(x_i) - b_i]}{h_i}, \quad i = 0, 1, 2 \ldots n - 1. \]

Note from the clamped condition, we have \( b_0 = 0 \) and then we can obtain \( b_i \) and then \( c_i \), \( i = 0, 1, 2, \ldots, n - 1. \)