Kinematics

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Kinematics of Serial Robots

• We know how to describe the transformation of a single rigid object w.r.t. a single frame

• If we have many rigid object in serial connection,

how to express and derive their spatial relations?
Overview – Robot Kinematics

- Forward Kinematics
  - Planar Robotic Systems, Representation of Serial Robots, Open Polygon Model, Denavit-Hartenberg Representation, Singularities

- Inverse Kinematics
  - Kinematic Decoupling, Inverse Position: Geometric Approach, Inverse Orientation

- Kinematics in a Nut Shell
For industrial robots, the main concern is the position and orientation of the end-effector or the attached tool.
A 2D example

- Tool Center Position (TCP) of a Planar Robotic Manipulator

\[ X = d + a \cos A + b \cos B + c \cos C \]
\[ Y = e + a \sin A + b \sin B + c \sin C \]
Open Polygon Representation

- Homogeneous transformations can be applied to all joints to get the end effector / tool position. However ...
  - The transformation matrix depends on how the coordinate systems are set up and how the structural parameters are defined.
  - Hence, how to make sure two people can develop same transformation matrices for the same robot?
Step 1: Assign local reference frame for each joint (z and x axes)

- Every coordinate frame is established following three rules:
  - The $z_{i-1}$ axis lies along the axis of motion of the $i$th joint
Step 1: Assign local reference frame for each joint (z and x axes)

- Every coordinate frame is established following three rules:
  - The $x_i$ axis is normal to the $z_{i-1}$ axis, and points away from it to the $z_i$ axis.
Step 1: Assign local reference frame for each joint (z and x axes)

- Every coordinate frame is established following three rules:
  - The $x_i$ axis forms the common perpendicular between the $z_{i-1}$ and $z_i$ axis
Choices for the base and end-effector frames

• Base Frame
  • You can choose any location for the coordinate frame $0$ in the robot base as long as the $z_0$ axis is aligned with the first joint.

• End-effector Frame
  • The last coordinate frame ($n$th frame) can be placed anywhere in the tool or end effector, as long as the $x_n$ axis is normal to $z_{n-1}$ axis.
Step 2: Determine the D-H parameters

- Relative pose between rigid bodies
  - Position + Orientation

- How many parameters do you need to fully specify their relative pose?
Joint Angle

- $\theta_i$ is the joint angle from the $x_{i-1}$ to the $x_i$ axis about the $z_{i-1}$ axis using the right hand rule.
• $d_i$ is the offset distance from the origin of the $(i - 1)$th coordinate frame to the intersection of the $z_{i-1}$ axis with the $x_i$ axis along the $z_{i-1}$ axis.

Distance between two X-axes
• $a_i$ is the distance from the intersection of the $z_{i-1}$ axis with the $x_i$ axis to the origin of the $i$th frame along the $x_i$ axis (the shortest distance between the $z_{i-1}$ and $z_i$ axes).
• $\alpha_i$ is the twisted angle from the $z_{i-1}$ axis to the $z_i$ axis about the $x_i$ axis (using the right-hand rule).
DH parameters

\( d \) is the depth along the previous joint's \( z \) axis.
Step 3: Specify the transformation matrix

\[
A_{i-1}^i = \text{Rot}(z, \theta_i) \cdot \text{Trans}(0, 0, d_i) \cdot \text{Trans}(a_i, 0, 0) \cdot \text{Rot}(x, \alpha_i)
\]

\[
= \begin{bmatrix}
    c_{\theta_i} & -s_{\theta_i} & 0 & 0 & 1 & 0 & 0 & 0 & a_i & 1 & 0 & 0 & 0 \\
    s_{\theta_i} & c_{\theta_i} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 & 1 & d_i & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

Complete transformation from base frame to tool frame:

\[
T_0^6 = A_0^1 A_1^2 A_2^3 A_3^4 A_4^5 A_5^6
\]
Vectors in the Transformation Matrix

- Recall:

\[ H = \begin{bmatrix} n_x & s_x & a_x & d_x \\ n_y & s_y & a_y & d_y \\ n_z & s_z & a_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n & s & a & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

- \( n \) = normal direction
- \( s \) = sliding direction
- \( a \) = approach direction
Example: A planar robot
Example: A planar robot

\[ \begin{array}{|c|c|c|c|c|} 
\hline
\# & \theta & d & a & \alpha \\
\hline
1 & \theta_1 & 0 & a_1 & 0 \\
2 & \theta_2 & 0 & a_2 & 0 \\
\hline
\end{array} \]
\[
T_{0}^{1} = A_{0}^{1} = \text{Rot}(z, \theta_{1}) \cdot \text{Trans}(0,0,d_{1}) \cdot \text{Trans}(a_{1},0,0) \cdot \text{Rot}(x, \alpha_{1})
\]

\[
\begin{bmatrix}
c_{\theta_{1}} & -s_{\theta_{1}} c_{\alpha_{1}} & s_{\theta_{1}} s_{\alpha_{1}} & a_{1} c_{\theta_{1}} \\
s_{\theta_{1}} c_{\alpha_{1}} & c_{\theta_{1}} c_{\alpha_{1}} & -c_{\theta_{1}} s_{\alpha_{1}} & a_{1} s_{\theta_{1}} \\
0 & s_{\alpha_{1}} & c_{\alpha_{1}} & d_{1} \\
0 & 0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
c_{\theta_{1}} & -s_{\theta_{1}} & 0 & a_{1} c_{\theta_{1}} \\
s_{\theta_{1}} c_{\theta_{1}} & 0 & a_{1} s_{\theta_{1}} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>#</th>
<th>(\theta)</th>
<th>(d)</th>
<th>(a)</th>
<th>(\alpha)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\theta_{1})</td>
<td>0</td>
<td>(a_{1})</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>(\theta_{2})</td>
<td>0</td>
<td>(a_{2})</td>
<td>0</td>
</tr>
</tbody>
</table>
\[
T_1^2 = A_1^2 = \text{Rot}(z, \theta_2) \cdot \text{Trans}(0, 0, d_2) \cdot \text{Trans}(a_2, 0, 0) \cdot \text{Rot}(x, \alpha_2)
\]

\[
= \begin{bmatrix}
  c_{\theta_2} & -s_{\theta_2}c_{\alpha_2} & s_{\theta_2}s_{\alpha_2} & a_2c_{\theta_2} \\
  s_{\theta_2}c_{\alpha_2} & c_{\theta_2} & -s_{\theta_2}s_{\alpha_2} & a_2s_{\theta_2} \\
  0 & s_{\alpha_2} & c_{\alpha_2} & d_2 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\times
\begin{bmatrix}
  c_{\theta_2} & -s_{\theta_2} & 0 & a_2c_{\theta_2} \\
  s_{\theta_2} & c_{\theta_2} & 0 & a_2s_{\theta_2} \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>#</th>
<th>\theta</th>
<th>d</th>
<th>a</th>
<th>\alpha</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>\theta_1</td>
<td>0</td>
<td>a_1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>\theta_2</td>
<td>0</td>
<td>a_2</td>
<td>0</td>
</tr>
</tbody>
</table>
\[ T_0^1 = A_0^1 \]

\[ T_0^2 = A_0^1 A_1^2 = \begin{bmatrix}
    c_{\theta_1} c_{\theta_2} - s_{\theta_1} s_{\theta_2} & -c_{\theta_1} s_{\theta_2} - s_{\theta_1} c_{\theta_2} & 0 & a_1 c_{\theta_1} + a_2 \left( c_{\theta_1} c_{\theta_2} - s_{\theta_1} s_{\theta_2} \right) \\
    s_{\theta_1} c_{\theta_2} + c_{\theta_1} s_{\theta_2} & c_{\theta_1} c_{\theta_2} - s_{\theta_1} s_{\theta_2} & 0 & a_1 s_{\theta_1} + a_2 \left( s_{\theta_1} c_{\theta_2} + c_{\theta_1} s_{\theta_2} \right) \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix} \]

\[ A_0^1 = \begin{bmatrix}
    c_{\theta_1} & -s_{\theta_1} & 0 & a_1 c_{\theta_1} \\
    s_{\theta_1} & c_{\theta_1} & 0 & a_1 s_{\theta_1} \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix} \]

\[ A_1^2 = \begin{bmatrix}
    c_{\theta_2} & -s_{\theta_2} & 0 & a_2 c_{\theta_2} \\
    s_{\theta_2} & c_{\theta_2} & 0 & a_2 s_{\theta_2} \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix} \]
<table>
<thead>
<tr>
<th>Link</th>
<th>$d_i$</th>
<th>$a_i$</th>
<th>$\alpha_i$</th>
<th>$\theta_i$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$-90\degree$</td>
<td>$\theta_1^*$</td>
</tr>
<tr>
<td>2</td>
<td>$d_2$</td>
<td>0</td>
<td>$90\degree$</td>
<td>$\theta_2^*$</td>
</tr>
<tr>
<td>3</td>
<td>$d_3^*$</td>
<td>0</td>
<td>0</td>
<td>$\theta_3^*$</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>$-90\degree$</td>
<td>$\theta_4^*$</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>$90\degree$</td>
<td>$\theta_5^*$</td>
</tr>
<tr>
<td>6</td>
<td>$d_6$</td>
<td>0</td>
<td>0</td>
<td>$\theta_6^*$</td>
</tr>
</tbody>
</table>
Example: A six-DOF articulate robot

<table>
<thead>
<tr>
<th>#</th>
<th>( \theta )</th>
<th>( d )</th>
<th>( a )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \theta_1 )</td>
<td>0</td>
<td>0</td>
<td>90</td>
</tr>
<tr>
<td>2</td>
<td>( \theta_2 )</td>
<td>0</td>
<td>( a_2 )</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>( \theta_3 )</td>
<td>0</td>
<td>( a_3 )</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>( \theta_4 )</td>
<td>0</td>
<td>( a_4 )</td>
<td>(-90)</td>
</tr>
<tr>
<td>5</td>
<td>( \theta_5 )</td>
<td>0</td>
<td>0</td>
<td>90</td>
</tr>
<tr>
<td>6</td>
<td>( \theta_6 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
\( T_0^1 = A_0^1 A_1^2 A_2^3 A_3^4 A_4^5 A_5^6 \)

\[
\begin{bmatrix}
  c_1\left(c_{234} c_5 c_6 - s_{234} s_6\right) & c_1\left(-c_{234} c_5 c_6 - s_{234} c_6\right) & c_1 \left(c_{234} s_5\right) & c_1 \left(c_{234} a_4 + c_2 a_3 + c_2 a_2\right) \\
  -s_1 s_5 c_6 & +s_1 s_5 s_6 & +s_1 c_5 & \\
  s_1\left(c_{234} c_5 c_6 - s_{234} s_6\right) & s_1\left(-c_{234} c_5 c_6 - s_{234} c_6\right) & s_1 \left(c_{234} s_5\right) & s_1 \left(c_{234} a_4 + c_2 a_3 + c_2 a_2\right) \\
  +c_1 s_5 s_6 & -c_1 s_5 s_6 & -c_1 c_5 & \\
  s_{234} c_5 c_6 + c_{234} s_6 & -s_{234} c_5 c_6 + c_{234} c_6 & s_{234} s_5 & s_{234} a_4 + s_2 a_3 + s_2 a_2 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]
Singularities

- There are 3 common singularities with serial robotics systems
  - Wrist alignment – joint 4 and 6 – collinear axis
  - Elbow singularity - Out-of-reach
  - Alignment singularity – wrist is as close to joint 1 as it can get
Degeneracy

- Degeneracy = Robot loses 1 DOF
  - Physical Limits
  - 2 similar joints become collinear
  - Determinant of position matrix = zero

- Reduced dexterity
  - Impossible to orient end effector at a desired orientation, at the limits of robots workspace.
Example: Spherical wrist singularity

- A spherical wrist
  - A singular configuration when the vectors $z_3$ and $z_5$ are linearly dependent.
  - The axes $z_3$ and $z_5$ are collinear, which happens when $\theta_5 = 0$ or $\pi$. 
Example: Spherical wrist singularity

- Unavoidable singularity for sphere wrist, unless …
  - The wrist is designed in such a way as not to permit this alignment.

- Not limited to a spherical wrist
  - If any two revolute joint axes become collinear a singularity results.
Wrist singularity
Gimbal Lock
2D Elbow Singularities

- The robot arm has two joints
- The joint space has 2 dimensions

- Theoretically, any position within the robot workspace is reachable by the end effector. However ...
  - A singularity reduces the mobility of the robot.
  - This will occur in two configurations – what are they?
2D Elbow Singularities

\[ \theta_2 = \begin{array}{ll}
0 & \text{Arm fully extended} \\
\pi & \text{Arm fully retracted}
\end{array} \]
2D Elbow singularity

\[ t = 0.01 \]
\[ u = [-6.046, -3.465,] \]
2D Elbow Singularities

$t = 0.01$
$u = [-6.046, -3.465]$
Singularity due to aligned links

Like the 2D case we just saw, there are two singularities due to the parallel $Z_1$ and $Z_2$ axes.
3D example

• Singularity due to aligned rotational axes
  • If the wrist center intersects with the axis of the base rotation, $z_0$, then there are an infinite number of solutions to the inverse kinematic equations.
  • In other words, any value of $\theta_1$ will produce the same wrist position. We have again lost a degree of freedom...
3D example

- Singularity due to reach limit
  - There are workspace volumes (shown in purple) where the end of arm tooling cannot reach.
• Singularity due to self collision
  • There are also configurations where the arm will collide with itself (another form of singularity).
Singularities of the ABB robot

Singularité d'épaule / Shoulder singularity
End