**Derivation of the Natural Cubic Spline**

Suppose we have \( a = x_0 < \ldots < x_n = b \) and \( y_0, \ldots, y_n \). A cubic interpolating spline for these data is a function \( S(x) \) that is twice continuously differentiable on \([a,b]\), satisfies \( S(x_i) = y_i \) for \( i = 0, \ldots, n \), and is such that \( S(x) = S_i(x) \), a cubic polynomial, for \( x \) in \([x_i, x_{i+1}]\) and \( i = 0, \ldots, n-1 \). The following derivation shows how to construct the cubic “pieces” \( S_i(x) \), \( i = 0, \ldots, n-1 \), using the “natural” end conditions \( S''(x_0) = S''(x_n) = 0 \).

**Notation:** \( \Delta x_i = x_{i+1} - x_i \) and \( \Delta y_i = y_{i+1} - y_i \) for \( i = 0, \ldots, n-1 \).

Suppose

\[
S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3, \quad i = 0, \ldots, n-1.
\]

We have two interpolation conditions on \( S_i(x) \): \( S_i(x_i) = S(x_i) = y_i \) and \( S_i(x_{i+1}) = S(x_{i+1}) = y_{i+1} \). The first immediately gives \( a_i = y_i \). With this, the second gives

\[
y_i + b_i \Delta x_i + c_i \Delta x_i^2 + d_i \Delta x_i^3 = y_{i+1}.
\]

Subtracting \( y_i \) from both sides and dividing by \( \Delta x_i \), we obtain

\[
b_i + c_i \Delta x_i + d_i \Delta x_i^2 = \frac{\Delta y_i}{\Delta x_i}, \quad i = 0, \ldots, n-1. \tag{1}
\]

The differentiability conditions are that \( S'(x) \) and \( S''(x) \) exist and are continuous throughout \([a,b]\). These conditions require that \( S'_i(x_{i+1}) = S'_{i+1}(x_{i+1}) \) and \( S''_i(x_{i+1}) = S''_{i+1}(x_{i+1}) \) for \( i = 0, \ldots, n-2 \). With \( S'_i(x) = b_i + 2c_i(x - x_i) + 3d_i(x - x_i)^2 \) for each \( i \), the first requirement becomes

\[
b_i + 2c_i \Delta x_i + 3d_i \Delta x_i^2 = b_{i+1}, \quad i = 0, \ldots, n-2. \tag{2}
\]

With \( S''_i(x) = 2c_i + 6d_i(x - x_i) \) for each \( i \), the second requirement becomes

\[
2c_i + 6d_i \Delta x_i = 2c_{i+1}, \quad \text{or just} \quad c_i + 3d_i \Delta x_i = c_{i+1}, \quad i = 0, \ldots, n-2.
\]

Since we also have \( S''(x_n) = S''_{n-1}(x_n) = 2c_{n-1} + 6d_{n-1} \Delta x_{n-1} \), we can define \( c_n = S''(x_n)/2 \) and extend the above relationship to \( i = n-1 \), obtaining

\[
c_i + 3d_i \Delta x_i = c_{i+1}, \quad i = 0, \ldots, n-1. \tag{3}
\]

Note that the natural end conditions \( S''(x_0) = S''(x_n) = 0 \) imply that \( c_0 = c_n = 0 \). Thus, we have \( n b_i \)'s, \( n-1 c_i \)'s, and \( n d_i \)'s, for a total of \( 3n - 1 \) coefficients remaining to be determined. Equations (1), (2), and (3) are \( 3n - 1 \) linear equations that these coefficients must satisfy.
We will manipulate (1), (2), and (3) to get a system of linear equations in the $c_i$'s alone. From (3), we obtain

$$d_i = \frac{c_{i+1} - c_i}{3\Delta x_i}, \quad i = 0, \ldots, n - 1. \quad (4)$$

Substituting this expression for $d_i$ in (1) and (2) yields

$$b_i + c_i \Delta x_i + \frac{\Delta x_i}{3} (c_{i+1} - c_i) = \frac{\Delta y_i}{\Delta x_i} \quad i = 0, \ldots, n - 1, \quad (5)$$

$$b_i + 2c_i \Delta x_i + \Delta x_i (c_{i+1} - c_i) = b_{i+1} \quad i = 0, \ldots, n - 2. \quad (6)$$

Subtracting (5) from (6) and rearranging, we get

$$b_{i+1} = \frac{\Delta y_i}{\Delta x_i} + c_i \Delta x_i + \frac{2}{3} \Delta x_i (c_{i+1} - c_i), \quad i = 0, \ldots, n - 2,$$

or, replacing $i$ by $i - 1$,

$$b_i = \frac{\Delta y_{i-1}}{\Delta x_{i-1}} + c_{i-1} \Delta x_{i-1} + \frac{2}{3} \Delta x_{i-1} (c_i - c_{i-1}), \quad i = 1, \ldots, n - 1. \quad (7)$$

Substituting this expression for $b_i$ in (5), we obtain

$$\Delta x_{i-1} c_{i-1} + 2(\Delta x_{i-1} + \Delta x_i) c_i + \Delta x_i c_{i+1} = 3 \left( \frac{\Delta y_i}{\Delta x_i} - \frac{\Delta y_{i-1}}{\Delta x_{i-1}} \right), \quad i = 1, \ldots, n - 1. \quad (7)$$

Since $c_0 = c_n = 0$, the equations (7) constitute a system of $n - 1$ equations in the $n - 1$ coefficients $c_1, \ldots, c_{n-1}$. Since $\Delta x_i > 0$ for each $i$, this system is strictly diagonally dominant (see Section 6.6 for the definition) and, as a consequence of Theorem 6.19, determines $c_1, \ldots, c_{n-1}$ uniquely. Having solved the system (7) for the $c_i$'s, we can obtain the $b_i$'s from (5) and the $d_i$'s from (4).

**Remark.** The system (7) has another very nice property in addition to strict diagonal dominance: It is tridiagonal, which means that, for each $i$, the only non-zero coefficients in the $i$th equation are those of $c_{i-1}$, $c_i$, and $c_{i+1}$. Tridiagonal systems can be solved very economically, specifically in a small multiple of $n$ arithmetic operations. In contrast, a general linear system, in which all coefficients may be nonzero, requires about $n^3/3$ multiplications and a similar number of additions and subtractions.