On the Null-Spaces of Partially Elliptic Operators of a Certain Type

JAMES A. LAVITA

University of Denver, Denver, Colorado 80210

AND

HOMER F. WALKER

University of Houston, Houston, Texas 77004

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1. Introduction

Recently, various authors have studied classes of nonelliptic partial differential operators arising in mathematical physics, for example [2, 3, 5]. One can show, among other results, that a coerciveness inequality holds for operators of the form

\[ A = \sum_{|\alpha| \leq m} a_{\alpha}(x) \frac{\partial^\alpha}{\partial x^\alpha} \]

(where \( x \in \mathbb{R}^n \), each \( a_{\alpha}(x) \) is a \( k \times k \) matrix, and the notation is standard multi-index notation) provided the following conditions are satisfied: If the symbol of \( A \) is given by

\[ a(x, \xi) = \sum_{|\alpha| \leq m} a_{\alpha}(x) \xi^\alpha, \]

then

(i) \( a(x, \xi) \) is of constant rank,

(ii) the null-space of \( a(x, \xi) \) is independent of \( x \).

Operators satisfying (i) and (ii) will be called partially elliptic operators. They are assumed to act on functions in the dense subset \( H_m(\mathbb{R}^n; \mathbb{C}^k) \) of \( L_2(\mathbb{R}^n; \mathbb{C}^k) \), denoted henceforth by \( H_m \) and \( L_2 \). The coerciveness of such operators holds on the set \( N(a)^{\perp} \cap H_m \), where \( N(a)^{\perp} \) is the orthogonal complement in \( L_2 \) of the set

\[ N(a) = \{ u : a(x, \xi) \hat{u}(\xi) = 0 \}. \]
In scattering theory, it is important to know something about the total null-space \(N(A)\) of such an operator \(A\) appearing in an evolution equation

\[u_t = Au.\]

Indeed, if \(A\) has no point spectrum other than zero, then it is to be expected that \(N(A)\) is precisely the set on which scattering does not take place. Clearly \(N(A) \supseteq N(a)\). To what extent can \(N(A)\) exceed \(N(a)\)? In the following, we show that if \(A\) belongs to a certain class of first-order partially elliptic operators, then \(N(A) = F_a \oplus N(a)\), where the dimension of \(F_a\) is finite and varies upper-semicontinuously with \(A\) in a certain sense. The development of these results closely parallels that of [6], using the coerciveness inequalities of [3]. Clearly, these results can be broadly extended in the spirit of [7] and [4]. However, in our opinion, energy would be better directed toward relaxing the condition (ii).

2. The Results

Let \(A_0 = \sum_1^n a_0(\partial / \partial x_j)\) be a first-order partial differential operator with symbol \(a_0(\xi) = \sum_1^n a_j(\xi)\). For each nonzero \(\xi \in \mathbb{R}^n\), let \(p(\xi)\) be the matrix of the projection onto the null-space of \(a_0(\xi)\), and let \(U(\xi)\) be a unitary matrix mapping the null-space of \(a_0(\xi)\) onto the orthogonal complement of the range of \(a_0(\xi)\). (We may assume that \(U(\xi)\), as well as \(p(\xi)\), is homogeneous of degree zero in \(\xi\).) Then, for each \(\xi \neq 0\),

\[b_0(\xi) = a_0(\xi) + |\xi| U(\xi) p(\xi)\]

is a nonsingular matrix, and the pseudo-differential operator \(B_0\) with symbol \(b_0(\xi)\) is an elliptic operator on \(L^2\), with domain \(H^1\). Define \(M(B_0, R) = \{ u \in H_1^1 : B_0 u(x) = 0 \text{ for } |x| \geq R\}\). We establish the following fundamental property of this set.

**Lemma 1.** Subsets of \(M(B_0, R)\) which are bounded in \(H^1\) are precompact in \(L^2\).

**Proof.** The proof is, in essence, the Fourier-transform proof of the Rellich Compactness Theorem (see [1]).

Suppose \(\{u_n\} \subseteq M(B_0, R)\) is a bounded sequence in \(H^1\). Set \(f_n = B_0 u_n\). Since \(\{f_n\}\) is an \(L^2\)-bounded sequence of functions with support in \(\{ x \in \mathbb{R}^n : |x| \leq R\}\), the Fourier transforms \(\hat{f}_n(\xi)\) are uniformly bounded with uniformly bounded derivatives in \(\mathbb{R}^n\). It follows from the Arzela–Ascoli Theorem that there exists a subsequence \(\{\hat{f}_{n_k}\}\) which converges uniformly on every compact subset of \(\mathbb{R}^n\).
Now for each \( \rho > 0 \), one has

\[
\| u_{nj} - u_{nk} \|^2 = \int_{|\xi| > \rho} |b_0(\xi)^{-1} [f_{nj}(\xi) - f_{nk}(\xi)]|^2 \, d\xi \\
+ \int_{|\xi| < \rho} |b_0(\xi)^{-1} [f_{nj}(\xi) - f_{nj}(\xi)]|^2 \, d\xi \\
\leq C \rho^2 \| f_{nj} - f_{nk} \|^2 + C \int_{|\xi| < \rho} \frac{1}{|\xi|^2} |f_{nj}(\xi) - f_{nk}(\xi)|^2 \, d\xi
\]

where \( C = \sup_{|\xi| = 1} |b_0(\xi)^{-1}|^2 \). Since the sequence \( \{f_{nj}\} \) is bounded in \( L^2 \), the first term on the right may be made small by taking \( \rho \) large. If \( n \geq 3 \), then \( \int_{|\xi| < \rho} (1/|\xi|^n) \, d\xi < \infty \), and the uniform convergence of the functions \( f_{nj} \) implies that the second term on the right approaches zero as \( j, k \to \infty \). If \( n = 2 \), then \( f_{nj}(0) = 0 \) for each \( j \), since \( u_{nj}(\xi) = b_0(\xi)^{-1} f_{nj}(\xi) \) is in \( L^2 \). Since the functions \( f_{nj} \) have uniformly bounded derivatives, the second term of the right must approach zero as \( j, k \to \infty \) when \( n = 2 \). It follows that \( \{u_{nj}\} \) is Cauchy in \( L^2 \) for all \( n \geq 2 \), and the proof is complete.

We now define \( PE(A_0, R) \) to be the set of partial differential operators of the form

\[
A = \sum_{j=1}^{n} a_j(x) \frac{\partial}{\partial x_j} + b(x)
\]

which are such that

(i) \( a_j(x) = a_j \) and \( b(x) = 0 \) for \( |x| \geq R \),

(ii) for each \( x \) and \( \xi \) in \( \mathbb{R}^n \), the null-space of \( a(x, \xi) = \sum_{j=1}^{n} a_j(x) \xi_j + b(x) \) is the same as the null-space of \( a_0(\xi) \).

If \( A \) is an operator in \( PE(A_0, R) \) then the pseudodifferential operator \( B \), with symbol \( h(x, \xi) = a(x, \xi) + |\xi| \sum_{j=1}^{n} a_j(x) \xi_j + b(x) \) is elliptic. Then a standard coerciveness inequality

\[
\| u \|_1 \leq C[\| u \| + \| Bu \|] \tag{1}
\]

holds for \( B \) (see, for example, [3]). Since \( N(B) \subseteq M(B_0, R) \), it follows from this inequality and from Lemma 1 that \( \{u \in N(B): \| u \| = 1\} \) is compact. Hence \( N(B) \) is finite-dimensional.

**Theorem 1.** If \( A \) is in \( PE(A_0, R) \), then

\[
N(A) = N(a_0) \oplus F_A
\]

where \( F_A \) is finite-dimensional and \( N(a_0) = \{u \in L^2: a_0(\xi) \tilde{u}(\xi) = 0\} \).
Proof. Since $N(a_0) \subset N(A)$, one has
\[ N(A) = (N(A) \cap N(a_0)^{\perp}) \oplus (N(A) \cap N(a_0)) \]
\[ = F_A \oplus N(a_0) \]
where $F_A = N(A) \cap N(a_0)^{\perp}$. Since $F_A \subset N(B)$, $F_A$ must be finite-dimensional.

It will now be shown that the dimension of $F_A$ depends upper-semi-continuously on an operator $A$ in $PE(A_0, R)$.

**Lemma 2.** If $B$ is the pseudodifferential operator associated with an operator $A$ in $PE(A_0, R)$, then there exists a $C > 0$ for which
\[ \| Bu \| \geq C \| u \| \]
for $u$ in $M(B_0, R) \cap N(B)^{\perp}$.

Proof. Suppose there exists no such constant. Then one can find a sequence \( \{u_n\} \subset M(B_0, R) \cap N(B)^{\perp} \) such that \( \| u_n \| = 1 \) and \( \| Bu_n \| \to 0 \). The inequality (1) implies that \( \{u_n\} \) is bounded in $H_1$, and so, by Lemma 1, there is a subsequence \( \{u_{n_j}\} \) which is Cauchy in $L_0$. Since $B$ is a closed operator, \( \lim_{j \to \infty} u_{n_j} = u_0 \) must be in $N(B)$. Since $u_0$ is the limit of functions in $N(B)^{\perp}$ having norm 1, this is a contradiction.

Now if $A$ and $\tilde{A}$ are two operators in $PE(A_0, R)$, it follows from the boundedness of the coefficients of these operators that there exists an $\epsilon > 0$ for which
\[ \|(A - \tilde{A}) u\| \leq \epsilon \| u \| \] (2)
for all $u$ in $H_1$. In particular, $\epsilon$ may be taken small if the coefficients of $\tilde{A}$ are near those of $A$ uniformly in $\mathbb{R}^n$.

**Theorem 2.** If $A$ is an operator in $PE(A_0, R)$, then there exists an $\epsilon_A > 0$ such that
\[ \text{dimension } F_{\tilde{A}} \leq \text{dimension } F_A \]
for every operator $\tilde{A}$ in $PE(A_0, R)$ satisfying \( \|(A - \tilde{A}) u\| \leq \epsilon_A \| u \|_1 \) for all $u \in H_1$.

Proof. Given $A$ and $\tilde{A}$ in $PE(A_0, R)$, let $B$ and $\tilde{B}$ be the associated pseudodifferential operators. Then by Lemma 2 and the inequalities (1) and (2), there exist constants $C$ and $\epsilon$ such that
\[ \| u \|_1 \leq C \| Bu \| \leq C \| Bu \| + C \| (B - B) u \| \]
\[ = C \| \tilde{B} u \| + C \| \tilde{A} u \| \]
\[ \leq C \| \tilde{B} u \| + C \epsilon \| u \|_1 \]
for all \( u \in M(B_0, R) \cap N(B)^\perp \). If \( \epsilon \leq \epsilon_A < 1/C \), then there exists a different constant \( C \), independent of \( \epsilon \), for which

\[
\| u \|_1 \leq C \| Bu \|
\]

for all \( u \in M(B_0, R) \cap N(B)^\perp \).

Suppose \( \tilde{A} \) is an operator in \( PE(A_0, R) \) with \( \epsilon \leq \epsilon_A \) as above, and suppose that \( \dim F_{\tilde{A}} > \dim F_A \). Then there exists a nonzero \( u_0 \) in \( F_{\tilde{A}} \cap F_A^\perp \). It follows that \( u_0 \in M(B_0, R) \cap N(B)^\perp \), and so \( \| u_0 \|_1 \leq C \| Bu_0 \| = 0 \), which is a contradiction.

References